

PURPLE COMET MATH MEET April 2010

HIGH SCHOOL - SOLUTIONS

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Problem 1

Let x satisfy $(6x + 7) + (8x + 9) = (10 + 11x) + (12 + 13x)$. There are relatively prime positive integers so that $x = -\frac{m}{n}$. Find $m + n$.

Answer: 8

The equation simplifies to $(6 + 8)x + (7 + 9) = (10 + 12) + (11 + 13)x$ or $14x + 16 = 22 + 24x$. Thus, $-10x = 6$ and $x = -\frac{3}{5}$. The requested sum is $3 + 5 = 8$.

Problem 2

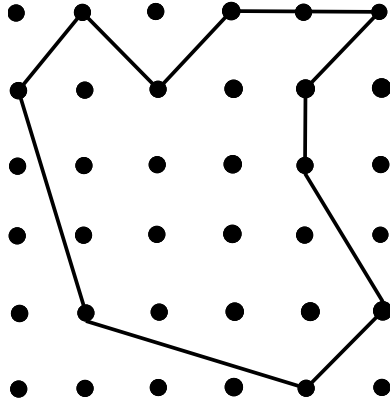
The prime factorization of $12 = 2 \cdot 2 \cdot 3$ has three prime factors. Find the number of prime factors in the factorization of $12! = 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$.

Answer: 19

$12! = 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = (2 \cdot 2 \cdot 3) \cdot 11 \cdot (2 \cdot 5) \cdot (3 \cdot 3) \cdot (2 \cdot 2 \cdot 2) \cdot 7 \cdot (2 \cdot 3) \cdot 5 \cdot (2 \cdot 2) \cdot 3 \cdot 2 \cdot 1$
which is a total for 19 prime factors.

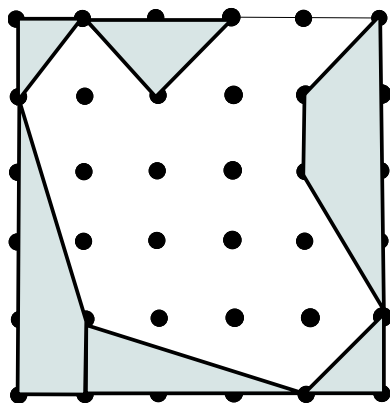
Problem 3

The grid below contains six rows with six points in each row. Points that are adjacent either horizontally or vertically are a distance two apart. Find the area of the irregularly shaped ten sided figure shown.



Answer: 66

The entire grid forms a square with side length 10, so the grid has area 100. The figure shown, thus, has area 100 minus the areas of two trapezoids and four right triangles shaded in the diagram shown below. Both trapezoids have bases of length 2 and 8 with height 2, so each has area $\frac{2+8}{2} \cdot 2 = 10$. The largest right triangle has legs with lengths 2 and 6, so it has area $\frac{2 \cdot 6}{2} = 6$. Two small right triangles each have two legs of length 2, so each has area $\frac{2 \cdot 2}{2} = 2$. The last right triangle has a hypotenuse of length 4 and an altitude to the hypotenuse of length 2, so its area is $\frac{2 \cdot 4}{2} = 4$. Therefore, the area of the ten sided figure is $100 - (10 + 10 + 6 + 2 + 2 + 4) = 100 - 34 = 66$.



Problem 4

Sally's salary in 2006 was \$37,500. For 2007 she got a salary increase of x percent. For 2008 she got another salary increase of x percent. For 2009 she got a salary decrease of $2x$ percent. Her 2009 salary is \$34,825. Suppose instead, Sally had gotten a $2x$ percent salary decrease for 2007, an x percent salary increase for 2008, and an x percent salary increase for 2009. What would her 2009 salary be then?

Answer: 34825

Let $y = \frac{x}{100}$. Sally's 2009 salary is given by $37000(1+y)(1+y)(1-2y) = 34825$. If the salary increases and decreases are reversed, her 2009 salary would be $37000(1-2y)(1+y)(1+y)$ which is still equal to 34825 since multiplication is a commutative operation. This happens if x is approximately 13.4144 percent.

Problem 5

If a and b are positive integers such that $a \cdot b = 2400$, find the least possible value of $a + b$.

Answer: 98

The number 2400 has 36 positive integer divisors. One can check all 18 pairs of these divisors, although the minimum sum will occur when the pair of divisors are closest to $\sqrt{2400}$ which is between 48 and 49. The pairs of divisors closest to this is 48 and 50 whose sum is 98 which is the least possible.

Problem 6

Evaluate the sum $1 + 2 - 3 + 4 + 5 - 6 + 7 + 8 - 9 + \dots + 208 + 209 - 210$.

Answer: 7245

Grouping the numbers in the sum in groups of three gives

$$1 + 2 - 3 + 4 + 5 - 6 + 7 + 8 - 9 + \dots + 208 + 209 - 210 =$$

$$(1 + 2 - 3) + (4 + 5 - 6) + (7 + 8 - 9) + \dots + (208 + 209 - 210) = 0 + 3 + 6 + \dots + 207 = 3(0 + 1 + 2 + \dots + 69).$$

Since the sum of the first n positive integers is given by $\frac{n(n+1)}{2}$, the desired sum is $3 \cdot \frac{69 \cdot 70}{2} = 7245$.

Problem 7

Find the sum of the digits in the decimal representation of the number $5^{2010} \cdot 16^{502}$.

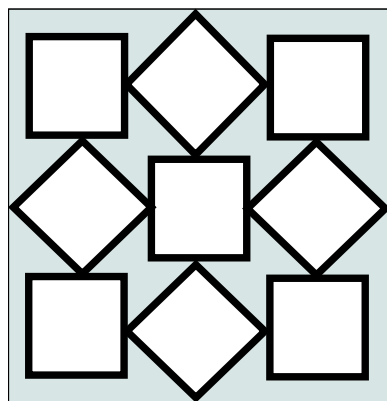
Answer: 7

$5^{2010} \cdot 16^{502} = 5^{2010} \cdot (2^4)^{502} = 5^{2010} \cdot 2^{2008} = 5^2 \cdot 10^{2008}$ which is the number 25 followed by 2008 zeros.

Thus, the sum of its digits is $2 + 5 = 7$.

Problem 8

The diagram below shows some small squares each with area 3 enclosed inside a larger square. Squares that touch each other do so with the corner of one square coinciding with the midpoint of a side of the other square. Find integer n such that the area of the shaded region inside the larger square but outside the smaller squares is \sqrt{n} .



Answer: 288

Each of the small squares has side length $\sqrt{3}$ and diagonal length $\sqrt{3}\sqrt{2} = \sqrt{6}$. Since the large square has sides equal to the sum of the lengths of two of the diagonals and one of the sides of the small squares, the area of the large square is $(\sqrt{3} + 2\sqrt{6})^2 = 3 + 4 \cdot \sqrt{3}\sqrt{6} + 4 \cdot 6 = 27 + 12\sqrt{2}$. Each small square has area 3, and there are nine small squares with a total area of 27. Thus, the desired area is $12\sqrt{2} = \sqrt{288}$. The requested answer is 288.

Problem 9

Find positive integer n so that $\frac{80-6\sqrt{n}}{n}$ is the reciprocal of $\frac{80+6\sqrt{n}}{n}$.

Answer: 64

If the two numbers are reciprocals, their product is $1 = \frac{80-6\sqrt{n}}{n} \cdot \frac{80+6\sqrt{n}}{n} = \frac{80^2-6^2n}{n^2}$ or $n^2 + 36n - 80^2 = 0$ which has solutions $n = \frac{-36 \pm \sqrt{36^2 + 4 \cdot 80^2}}{2} = \frac{-36 \pm 4\sqrt{9^2 + 40^2}}{2} = -18 \pm 2 \cdot 41$. Thus, the positive solution is $2 \cdot 41 - 18 = 64$.

Problem 10

The set S contains nine numbers. The mean of the numbers in S is 202. The mean of the five smallest of the numbers in S is 100. The mean of the five largest numbers in S is 300. What is the median of the numbers in S ?

Answer: 182

Let the median be m . The sum of all the numbers in S is $9 \cdot 202$. The sum of the five smallest numbers in S is $5 \cdot 100$. The sum of the five largest numbers in S is $5 \cdot 300$. Thus, $m = 5 \cdot 100 + 5 \cdot 300 - 9 \cdot 202 = 500 + 1500 - 1818 = 182$.

Problem 11

A jar contains one white marble, two blue marbles, three red marbles, and four green marbles. If you select two of these marbles without replacement, the probability that both marbles will be the same color is $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

Answer: 11

There are ten marbles in all, so there are $\binom{10}{2} = 45$ equally likely ways to select two marbles. There is no way to select two white marbles, $\binom{2}{2} = 1$ way to select two blue marbles, $\binom{3}{2} = 3$ ways to select two red marbles, and $\binom{4}{2} = 6$ ways to select two green marbles. Thus, the probability of selecting two marbles of the same color is $\frac{0+1+3+6}{45} = \frac{10}{45} = \frac{2}{9}$. The requested sum is $2 + 9 = 11$.

Problem 12

A good approximation of π is 3.14. Find the least positive integer d such that if the area of a circle with diameter d is calculated using the approximation 3.14, the error will exceed 1.

Answer: 51

The answer is the least positive integer d such that $\left(\frac{d}{2}\right)^2(\pi - 3.14) \geq 1$. That is, $d^2 \geq \frac{4}{\pi - 3.14} \approx \frac{4}{0.00159} \approx 2511$. The value of d is thus 51.

Problem 13

Let S be the set of all 10-term arithmetic progressions that include the numbers 4 and 10. For example, $(-2, 1, 4, 7, 10, 13, 16, 19, 22, 25)$ and $(10, 8\frac{1}{2}, 7, 5\frac{1}{2}, 4, 2\frac{1}{2}, 1, -\frac{1}{2}, -2, -3\frac{1}{2})$ are both members of S . Find the sum of all values of a_{10} for each $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}) \in S$, that is,
$$\sum_{(a_1, a_2, a_3, \dots, a_{10}) \in S} a_{10}.$$

Answer: 630

An arithmetic progression containing the terms 4 and 10 is completely determined by the positions in the progression of the 4 and 10. For each $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}) \in S$ there is exactly one $(b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}) \in S$ where the positions of the 4 and the 10 are reversed. For example, the sequence $(-2, 1, 4, 7, 10, 13, 16, 19, 22, 25)$ and the sequence $(16, 13, 10, 7, 4, 1, -2, -5, -8, -11)$ have their 4 and 10 terms in opposite positions. Then $(a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4, a_5 + b_5, a_6 + b_6, a_7 + b_7, a_8 + b_8, a_9 + b_9, a_{10} + b_{10})$ is also an arithmetic progression, but since two of its terms are $4 + 10 = 14$, it is a constant progression with all of its terms equal to 14. In particular, $a_{10} + b_{10} = 14$. It follows that mean of all tenth terms of progressions in S is $\frac{14}{2} = 7$, and the requested sum of all tenth terms is 7 times the number of elements in S . Since an element of S is completely determined by the placement of the terms 4 and 10, and there are ten possible placements of the term 4, and nine possible placements of the term 10 once the 4 has been placed, the size of S is $10 \cdot 9 = 90$. Thus, the requested sum is $7 \cdot 90 = 630$.

Problem 14

There are positive integers b and c such that the polynomial $2x^2 + bx + c$ has two real roots which differ by 30. Find the least possible value of $b + c$.

Answer: 126

The quadratic formula says that the two roots of the polynomial are $\frac{-b+\sqrt{b^2-8c}}{4}$ and $\frac{-b-\sqrt{b^2-8c}}{4}$. These two roots differ by $\frac{\sqrt{b^2-8c}}{2} = 30$ or $b^2 - 8c = 3600$. Since c is positive, the smallest value of b and c occur when $3600 + 8c$ is the smallest perfect square multiple of 8 exceeding 60^2 . This is $64^2 = 4096$ which occurs when $b = 64$ and $c = \frac{4096-3600}{8} = 62$. Thus, the least possible value of $b + c$ is $64 + 62 = 126$.

Problem 15

Find the smallest possible sum $a + b + c + d + e$ where a, b, c, d , and e are positive integers satisfying the conditions

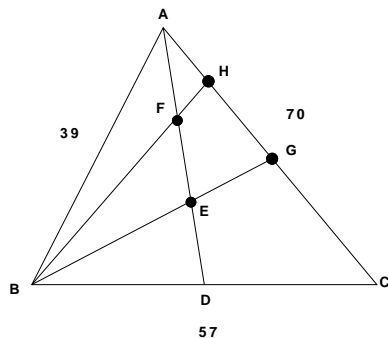
- each of the pairs of integers (a, b) , (b, c) , (c, d) , and (d, e) are **not** relatively prime
- all other pairs of the five integers **are** relatively prime.

Answer: 47

Since a and b are not relatively prime, there must be a prime number p_1 that divides both a and b . Similarly, there must be a prime number p_2 that divides both b and c , but since a and c are relatively prime, p_1 and p_2 are distinct. Similarly, there is a third prime p_3 that divides both c and d , and a fourth prime p_4 that divides both d and e , with all four primes being distinct. Thus, the smallest possible value for the sum $a + b + c + d + e$ is $p_1 + p_1 \cdot p_2 + p_2 \cdot p_3 + p_3 \cdot p_4 + p_4$. It follows that the four primes should be the smallest four prime numbers: 2, 3, 5, 7. It is also clear that p_1 and p_4 should be the largest of those four prime numbers. An assignment that minimizes the sum is $p_1 = 5, p_2 = 3, p_3 = 2, p_4 = 7$ resulting in the sum $5 + 5 \cdot 3 + 3 \cdot 2 + 2 \cdot 7 + 7 = 5 + 15 + 6 + 14 + 7 = 47$.

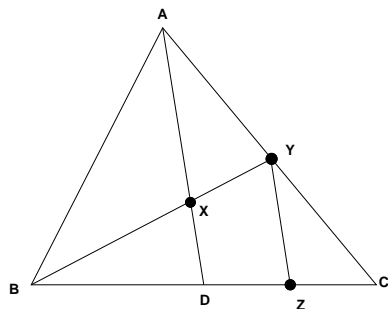
Problem 16

The triangle ABC has sides lengths $AB = 39$, $BC = 57$, and $CA = 70$ as shown. Median \overline{AD} is divided into three congruent segments by points E and F . Lines \overline{BE} and \overline{BF} intersect side \overline{AC} at points G and H , respectively. Find the distance from G to H .



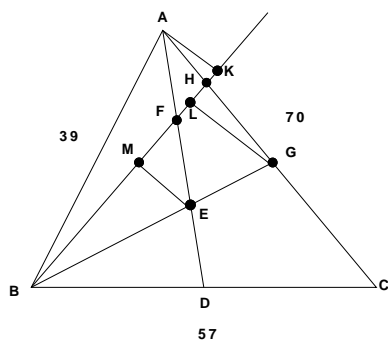
Answer: 21

More generally, let X be a point on median \overline{AD} , and let line \overline{BX} intersect side \overline{CA} at point Y as shown below. Let Z be a point on side \overline{BC} so that \overline{YZ} is parallel to \overline{XD} . Then triangles BDX and BZY are similar, and triangles YZC and ADC are similar. This implies that $\frac{BZ}{YZ} = \frac{BD}{DX}$ and $\frac{CZ}{YZ} = \frac{CD}{AD}$, so $\frac{BD}{DX} + \frac{CD}{AD} = \frac{BZ+CZ}{YZ} = \frac{BC}{YZ}$. Also, $\frac{YC}{YZ} = \frac{AC}{AD}$, so $YC = YZ \frac{AC}{AD} = \frac{BC}{\frac{BD}{DX} + \frac{CD}{AD}} \cdot \frac{AC}{AD} = \frac{BC}{\frac{BD}{DX} + \frac{CD}{AD}} \cdot \frac{AC}{AD} = \frac{2}{\frac{AD}{DX} + 1} \cdot AC$. Applying this when $DX = \frac{1}{3} \cdot AD$ shows that $GC = \frac{2}{3+1} \cdot 70 = 35$. Applying it when $DX = \frac{2}{3} \cdot AD$ shows that $HC = \frac{2}{\frac{3}{2}+1} \cdot 70 = 56$. Thus, $GH = 56 - 35 = 21$. Note that the solution does not depend on the lengths of \overline{AB} and \overline{BC} .



Alternatively, drop perpendiculars from A , G and E to \overline{BH} and denote the feet of these perpendiculars by K , L and M , respectively. Triangles AFB and BFE have equal areas since $\overline{AF} \equiv \overline{FE}$, and the corresponding heights are the same. These triangles share side \overline{BF} so the corresponding heights must be equal, $AK = EM$.

Triangles BEM and BGL are similar because they have two congruent angles, and thus $\frac{GL}{EM} = \frac{BG}{BE}$. Since E is one third the distance from D to A along a median, E is the centroid of triangle ABC , and $\frac{BG}{BE} = \frac{3}{2}$. Then the ratio of similarity must be the same for triangles LHG and KHA , thus $HG = \frac{3}{2}AH$. Since E is the centroid, \overline{BG} is a median, and $AG = 35$, $HG = \frac{3}{5} \cdot 35 = 21$.



Problem 17

Alan, Barb, Cory, and Doug are on the golf team; Doug, Emma, Fran, and Greg are on the swim team; and Greg, Hope, Inga, and Alan are on the tennis team. These nine people sit in a circle in random order. The probability that no two people from the same team sit next to each other is $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

Answer: 2521

Nine people can sit in a circle in $8!$ equally likely arrangements. For the arrangement to have members of the same team not sitting next to each other, Greg, who is on both the swim team and the tennis team, must sit next to both Barb and Cory. Similarly, Alan, who is on both the golf team and the tennis team, must sit next to both Emma and Fran, and Doug, who is on both the golf team and the swim team, must sit next to both Hope and Inga. For each of these groups of three people: $\{\text{Barb, Greg, Cory}\}$, $\{\text{Emma, Alan, Fran}\}$, $\{\text{Hope, Doug, Inga}\}$, there are two arrangements of the people in the group (for example, Barb, Greg, Cory and Cory, Greg, Barb). There are also $2! = 2$ arrangements of the three groups around the circle. It follows that there are $2 \cdot 2 \cdot 2 \cdot 2 = 16$ arrangements of the people so that no two members of the same team sit next to each other. The requested probability is $\frac{16}{8!} = \frac{16}{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{1}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3} = \frac{1}{2520}$. The requested sum is $1 + 2520 = 2521$.

Problem 18

When $4 \cos \theta - 3 \sin \theta = \frac{13}{3}$, it follows that $7 \cos 2\theta - 24 \sin 2\theta = \frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

Answer: 122

Because $3^2 + 4^2 = 5^2$, there is an angle ϕ so that $\cos \phi = \frac{4}{5}$ and $\sin \phi = \frac{3}{5}$. Recall the angle sum formula $\cos(\phi + \theta) = \cos \phi \cdot \cos \theta - \sin \phi \cdot \sin \theta$ and the double angle formulas $\cos 2\theta = 2 \cos^2 \theta - 1$ and $\sin 2\theta = 2 \sin \theta \cos \theta$. Thus, $\cos 2\phi = 2 \cos^2 \phi - 1 = 2 \left(\frac{4}{5}\right)^2 - 1 = \frac{7}{25}$ and $\sin 2\phi = 2 \sin \phi \cos \phi = 2 \frac{4}{5} \frac{3}{5} = \frac{24}{25}$.

From the given information one sees that $\cos(\phi + \theta) = \frac{4 \cos \theta - 3 \sin \theta}{5} = \frac{13}{15}$. Then,

$$\cos 2(\phi + \theta) = 2 \cos^2(\phi + \theta) - 1 = 2 \left(\frac{13}{15}\right)^2 - 1 = \frac{338}{225} - 1 = \frac{113}{225}. \text{ Also,}$$

$$\cos 2(\phi + \theta) = \cos 2\phi \cdot \cos 2\theta - \sin 2\phi \cdot \sin 2\theta = \frac{7}{25} \cos 2\theta - \frac{24}{25} \sin 2\theta = \frac{1}{25} \frac{m}{n}. \text{ It follows that } \frac{m}{n} = 25 \frac{113}{225} = \frac{113}{9}.$$

The requested sum is $113 + 9 = 122$.

Alternatively, set $x = \cos \theta$ and $y = \sin \theta$. Then it is known that $4x - 3y = \frac{13}{3}$ and $x^2 + y^2 = 1$, and, from the double angle formulas for $\sin \theta$ and $\cos \theta$, the requested value is

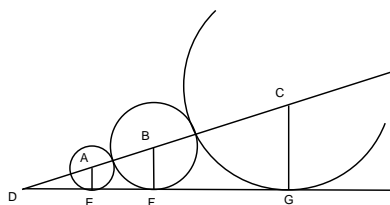
$$7 \cos 2\theta - 24 \sin 2\theta = 7(2x^2 - 1) - 24(2xy). \text{ But then}$$

$$7(2x^2 - 1) - 24(2xy) = 14x^2 - 7 - 48xy = -18(x^2 + y^2) + 2(4x - 3y)^2 - 7 = -18(1) + 2 \left(\frac{13}{3}\right)^2 - 7 = \frac{113}{9}.$$

Problem 19

The centers of the three circles A , B , and C are collinear with the center of circle B lying between the centers of circles A and C . Circles A and C are both externally tangent to circle B , and the three circles share a common tangent line. Given that circle A has radius 12 and circle B has radius 42, find the radius of circle C .

Answer: 147



Let circles A , B , and C have centers A , B , C and radii a , b , and c , and be tangent to their common tangent at points E , F , and G , respectively. Let D be the intersection of the common tangent and the line containing the centers of the circles. Then triangles ADE , BDF , and CDG are similar. It follows that

$\frac{a}{AD} = \frac{b}{BD} = \frac{b}{AD+a+b}$. This yields $\frac{AD}{a} = \frac{b+a}{b-a}$, so by similar triangles

$\frac{AD}{a} = \frac{CD}{c} = \frac{AD+a+2b+c}{c} = \frac{a\frac{b+a}{b-a} + a + 2b + c}{c}$. Solving for c gives

$$c = \frac{a\frac{b+a}{b-a} + a + 2b}{\frac{b+a}{b-a} - 1} = \frac{a(b+a) + (a+2b)(b-a)}{(b+a) - (b-a)} = \frac{b^2}{a} = \frac{42^2}{12} = 147.$$

Alternatively, note that there is a dilation centered at D that maps circle A to circle B . The constant of this dilation is $\frac{42}{12} = \frac{7}{2}$. The same dilation moves circle B to circle C , thus the radius of circle C is

$$\frac{7}{2} \cdot 42 = 147.$$

Problem 20

How many of the rearrangements of the digits 123456 have the property that for each digit, no more than two digits smaller than that digit appear to the right of that digit? For example, the rearrangement 315426 has this property because digits 1 and 2 are the only digits smaller than 3 which follow 3, digits 2 and 4 are the only digits smaller than 5 which follow 5, and digit 2 is the only digit smaller than 4 which follows 4.

Answer: 162

Begin with six open positions. To get an arrangement of digits having the desired property, choose a position to place the digit 6. The 6 can be placed in any of the 3 rightmost positions. Once the position of the 6 has been chosen, choose a position to place the digit 5. The 5 can be placed in any of the 3 rightmost positions not occupied by the 6. Continuing this way, the digits 6, 5, 4, and 3 can all be placed in one of 3 positions, the 3 rightmost positions which are left open. Finally, there will be 2 available positions to place the 2, and only one position left for the 1. Thus, the number of ways to choose an arrangement with the desired property is $3^4 \cdot 2 = 162$.

Problem 21

Let a be the sum of the numbers:

$$99 \times 0.9$$

$$999 \times 0.9$$

$$9999 \times 0.9$$

\vdots

$$999 \dots 9 \times 0.9$$

where the final number in the list is 0.9 times a number written as a string of 101 digits all equal to 9.

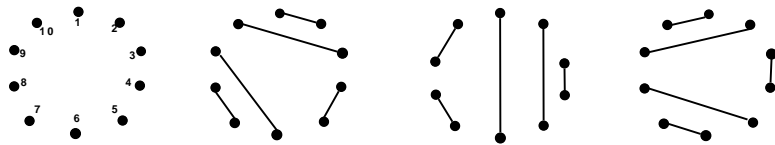
Find the sum of the digits in the number a .

Answer: 891

The list contains 100 numbers. The sum is $a = 0.9 \times (99 + 999 + 9999 + \dots + 999 \dots 9) = 0.9 \times [(100 - 1) + (1000 - 1) + (10000 - 1) + \dots + (1000 \dots 0 - 1)] = 0.9 \times \left[\sum_{n=0}^{101} (10^n - 1) - 9 \right] = \frac{9}{10} \cdot \left[\sum_{n=0}^{101} 10^n - 111 \right] = \frac{9}{10} \cdot \left[\frac{10^{102} - 1}{10 - 1} - 111 \right] = \frac{1}{10} [(10^{102} - 1) - 999] = 10^{101} - 100$. The decimal representation of this number is 101 digits long and consists of 99 nines followed by 2 zeros. The sum of the digits of this number is $99 \cdot 9 = 891$.

Problem 22

Ten distinct points are placed in a circle. All ten of the points are paired so that the line segments connecting the pairs do not intersect. In how many different ways can this pairing be done?



Answer: 42

More generally, let a_n be the number of ways one can draw n non-intersecting line segments between the pairings of $2n$ points on a circle. Clearly, $a_1 = 1$. If a_1 through a_n are known, a_{n+1} can be calculated as follows. Number the points in the circle in order from 1 to $2n + 2$. Point 1 will need to be paired with one of the n points labeled with an even number so that there will be an even number of points lying on each side of the line segment drawn from point 1. If point 1 is paired with point $2k$, then there are $2k - 2$ points left on one side of the segment, and $2n - 2k$ points left on the other side. It follows that there are $a_{k-1} \cdot a_{n-k}$ ways to pair the remaining points. This gives the recursive formula $a_{n+1} = \sum_{k=1}^{n+1} a_{k-1} \cdot a_{n-k}$ assuming that a_0 is defined to be 1. Thus,

$$a_2 = a_0 \cdot a_1 + a_1 \cdot a_0 = 2$$

$$a_3 = a_0 \cdot a_2 + a_1 \cdot a_1 + a_2 \cdot a_0 = 5$$

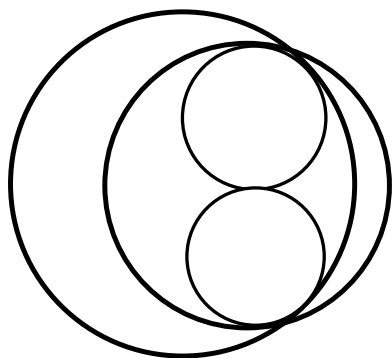
$$a_4 = a_0 \cdot a_3 + a_1 \cdot a_2 + a_2 \cdot a_1 + a_3 \cdot a_0 = 14$$

$$a_5 = a_0 \cdot a_4 + a_1 \cdot a_3 + a_2 \cdot a_2 + a_3 \cdot a_1 + a_4 \cdot a_0 = 42.$$

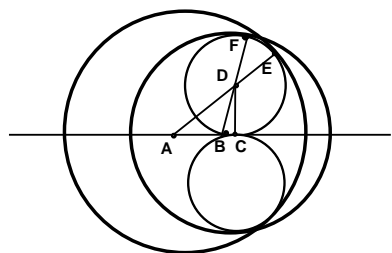
Thus, the answer is 42. These numbers are also the Catalan numbers $a_n = C(n) = \frac{\binom{2n}{n}}{n+1}$.

Problem 23

A disk with radius 10 and a disk with radius 8 are drawn so that the distance between their centers is 3. Two congruent small circles lie in the intersection of the two disks so that they are tangent to each other and to each of the larger circles as shown. The radii of the smaller circles are both $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.



Answer: 19



Let A be the center of the circle with radius 10, and B be the center of the circle with radius 8. The two small circles are tangent to each other and the line AB at point C . Let D be the center of one of the small circles, E be the point where that small circle is tangent to the circle centered at A , and F be the point where that small circle is tangent to the circle centered at B . The triangles ACD and BCD are right triangles with right angles at C . Let r be the radius of the small circles. Then $CD = r$. Since $AE = 10$, it follows from the Pythagorean Theorem that $AC = \sqrt{(10 - r)^2 - r^2}$. Since $BF = 8$, it follows from the Pythagorean Theorem that $BC = \sqrt{(8 - r)^2 - r^2}$. But $AC = BC + 3$, so $\sqrt{(10 - r)^2 - r^2} = \sqrt{(8 - r)^2 - r^2} + 3$. Square both sides to get $100 - 20r = 64 - 16r + 6\sqrt{(8 - r)^2 - r^2} + 9$. Solving for the radical term and squaring again yields $6\sqrt{(8 - r)^2 - r^2} = 27 - 4r$ and $36(64 - 16r) = 27^2 - 216r + 16r^2$. This simplifies to $16r^2 + 360r - 1575 = 0$ which has one positive solution at $r = \frac{-360 + \sqrt{360^2 + 4 \cdot 16 \cdot 1575}}{2 \cdot 16} = \frac{-360 + 24\sqrt{15^2 + 175}}{32} = \frac{-360 + 24 \cdot 20}{32} = \frac{15}{4}$. The requested sum is $15 + 4 = 19$.

Problem 24

Find the number of ordered pairs of integers (m, n) that satisfy $20m - 10n = mn$.

Answer: 24

Rewrite the condition as $mn - 20m + 10n = 0$ and then $mn - 20m + 10n - 200 = -200$ which factors as $(m + 10)(n - 20) = -200$. Thus, (m, n) satisfies the desired condition if the product of $m + 10$ and $n - 20$ is -200. The 12 positive integer divisors of 200 are 1, 2, 4, 5, 8, 10, 20, 25, 40, 50, 100, and 200. If $m + 10$ is any one of these positive integer divisors of 200 or the negative of any one of them, then setting $n = \frac{-200}{m+10} + 20$ lets (m, n) satisfy the required condition. Thus, there are 24 ordered pairs satisfying the condition.

Problem 25

Let x_1, x_2 , and x_3 be the three roots of the polynomial $x^3 + 3x + 1$. There are relatively prime positive integers m and n so that $\frac{m}{n} = \frac{x_1^2}{(5x_2+1)(5x_3+1)} + \frac{x_2^2}{(5x_1+1)(5x_3+1)} + \frac{x_3^2}{(5x_1+1)(5x_2+1)}$. Find $m + n$.

Answer: 10

First note that since $x^3 + 3x + 1 = (x - x_1)(x - x_2)(x - x_3)$, we get $x_1 + x_2 + x_3 = 0$, $x_1x_2 + x_1x_3 + x_2x_3 = 3$, and $x_1x_2x_3 = -1$. It also follows that

$0 = (x_1 + x_2 + x_3)^2 = x_1^2 + x_2^2 + x_3^2 + 2(x_1x_2 + x_1x_3 + x_2x_3)$ and $x_1^2 + x_2^2 + x_3^2 = -6$. Finally, since each x_j satisfies $x^3 + 3x + 1 = 0$, one gets that $x_1^3 + x_2^3 + x_3^3 = -3(x_1 + x_2 + x_3) - 3 = -3$.

Now, $\frac{x_1^2}{(5x_2+1)(5x_3+1)} + \frac{x_2^2}{(5x_1+1)(5x_3+1)} + \frac{x_3^2}{(5x_1+1)(5x_2+1)} = \frac{x_1^2(5x_1+1) + x_2^2(5x_2+1) + x_3^2(5x_3+1)}{(5x_1+1)(5x_2+1)(5x_3+1)} = \frac{5(x_1^3 + x_2^3 + x_3^3) + (x_1^2 + x_2^2 + x_3^2)}{5^3(x_1x_2x_3) + 5^2(x_1x_2 + x_1x_3 + x_2x_3) + 5(x_1 + x_2 + x_3) + 1} = \frac{5(-3) - 6}{125(-1) + 25(3) + 5(0) + 1} = \frac{-21}{-125 + 75 + 1} = \frac{21}{49} = \frac{3}{7}$. The requested sum is $3 + 7 = 10$.

Problem 26

In the coordinate plane a parabola passes through the points (7,6), (7,12), (18,19), and (18,48). The axis of symmetry of the parabola is a line with slope $\frac{r}{s}$ where r and s are relatively prime positive integers. Find $r + s$.

Answer: 71

First consider the parabola with equation $y = x^2$ and two parallel lines with equations $y = mx + b_1$ and $y = mx + b_2$ for constants m , $b_1 > 0$, and $b_2 > 0$. Note that the axis of symmetry of the parabola is the y axis. The first line intersects the parabola at two points with x coordinates $\frac{m \pm \sqrt{m^2 + 4b_1}}{2}$, and the second line at x coordinates $\frac{m \pm \sqrt{m^2 + 4b_2}}{2}$. It follows that the difference between the x coordinates of the intersection points to the left of the y axis is $\frac{\sqrt{m^2 + 4b_1} - \sqrt{m^2 + 4b_2}}{2}$ which is also the difference between the x coordinates of the intersection points to the right of the y axis. Since all parabolas are geometrically similar, this shows that when two parallel lines intersect a parabola, the difference in the distances from the axis of symmetry for two intersection points on one side of the axis is the same as the difference in distances from the axis of symmetry for the two intersection points on the other side of the axis.

Because the slope of the axis of symmetry will not change if the parabola is translated seven units in the negative x direction and 6 units in the negative y direction, it can be assumed that the given parabola passes through the four points (0,0), (0,6), (11,13), and (11,42). Assume that the axis of symmetry of the parabola has slope m . Then there are constants a and b so that the line parallel to the axis which passes through (11,13) has equation $y = mx + a$, and the line parallel to the axis which passes through (11,42) has equation $y = mx + b$. It follows that $13 = 11m + a$ and $42 = 11m + b$ which implies that $b - a = 42 - 13 = 29$. The result of the previous paragraph says that the lines $y = mx + b$ and $y = mx + 6$ must be the same distance apart as the lines $y = mx$ and $y = mx + a$. Thus, $b - 6 = 0 - a$ or $a + b = 6$. This lets us solve for b to get $b = \frac{35}{2}$. So, the line parallel to the axis of symmetry which passes through the point (11,42) also passes through the point $(0, b = \frac{35}{2})$. Conclude that the slope m of this line, and thus the slope of the axis of symmetry, is $\frac{42 - \frac{35}{2}}{11} = \frac{49}{22}$. The requested sum is $49 + 22 = 71$.

Problem 27

Let a and b be real numbers satisfying $2(\sin a + \cos a) \sin b = 3 - \cos b$. Find $3 \tan^2 a + 4 \tan^2 b$.

Answer: 35

Apply the Cauchy-Schwartz inequality to the vectors $(2(\sin a + \cos a), 1)$ and $(\sin b, \cos b)$ to get that $[4(\sin a + \cos a)^2 + 1](\sin^2 b + \cos^2 b) \geq [2(\sin a + \cos a) \sin b + \cos b]^2 = 3^2$ which simplifies to $[4(\sin^2 a + 2 \sin a \cos a + \cos^2 a) + 1](\sin^2 b + \cos^2 b) = 5 + 4 \sin 2a \geq 9$. Thus, $\sin 2a \geq 1$, and it follows that $\sin 2a = 1$. Use the identity $\tan \frac{x}{2} = \frac{\sin x}{1 + \cos x}$ to conclude that $\tan a = \frac{1}{1+0} = 1$. Since equality occurs in the Cauchy-Schwartz inequality only when the vectors are parallel, conclude that $\frac{\sin b}{\cos b} = \frac{2(\sin a + \cos a)}{1}$ and $\tan^2 b = 4(\sin^2 a + 2 \sin a \cos a + \cos^2 a) = 4(1 + \sin 2a) = 8$. Now $3 \tan^2 a + 4 \tan^2 b = 3(1) + 4(8) = 35$.

Problem 28

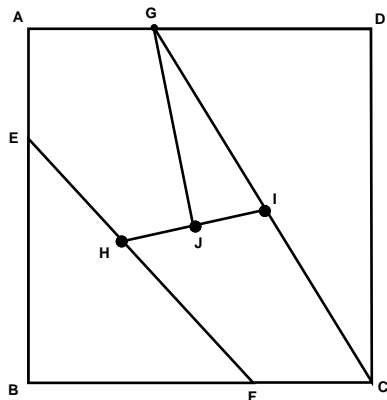
There are relatively prime positive integers p and q such that $\frac{p}{q} = \sum_{n=3}^{\infty} \frac{1}{n^5 - 5n^3 + 4n}$. Find $p + q$.

Answer: 97

The sum $\sum_{n=3}^{\infty} \frac{1}{n^5 - 5n^3 + 4n} = \sum_{n=3}^{\infty} \frac{1}{(n-2)(n-1)n(n+1)(n+2)} = \frac{1}{4} \sum_{n=3}^{\infty} \frac{(n+2) - (n-2)}{(n-2)(n-1)n(n+1)(n+2)} = \frac{1}{4} \sum_{n=3}^{\infty} \left(\frac{1}{(n-2)(n-1)n(n+1)} - \frac{1}{(n-1)n(n+1)(n+2)} \right)$. This last summation telescopes leaving $\frac{1}{4} \frac{1}{(3-2)(3-1)3(3+1)} = \frac{1}{4 \cdot 1 \cdot 2 \cdot 3 \cdot 4} = \frac{1}{96}$. The requested sum is $1 + 96 = 97$.

Problem 29

Square $ABCD$ is shown in the diagram below. Points E , F , and G are on sides \overline{AB} , \overline{BC} and \overline{DA} , respectively, so that lengths BE , BF , and DG are equal. Points H and I are the midpoints of segments \overline{EF} and \overline{CG} , respectively. Segment \overline{GJ} is the perpendicular bisector of segment \overline{HI} . The ratio of the areas of pentagon $AEHJG$ and quadrilateral $CIHF$ can be written as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.



Answer: 175

Without loss of generality, let the square have side length 18. Set up coordinate axes so that \overline{BC} is the x-axis and \overline{AB} is the y-axis. Then the coordinates of point $A = (0, 18)$, $B = (0, 0)$, $C = (18, 0)$, and $D = (18, 18)$. Let the coordinates of point E be $(0, y)$. Then the coordinates of point $F = (y, 0)$, $G = (18 - y, 18)$, $H = (\frac{y}{2}, \frac{y}{2})$, $I = (18 - \frac{y}{2}, 9)$, and $J = (9, \frac{18+y}{4})$. The line \overline{GJ} is perpendicular to the line \overline{HI} , so the slope of $\overline{GJ} = \frac{54-y}{9-y}$ must be the negative reciprocal of the slope of $\overline{HI} = \frac{9-\frac{y}{2}}{18-y} = \frac{1}{2}$. It follows that $\frac{54-y}{4(9-y)} = -2$ so $y = 14$.

The areas of pentagon $AEHJG$ and quadrilateral $CIHF$ can be calculated from the coordinates of their vertices $A = (0, 18)$, $E = (0, 14)$, $H = (7, 7)$, $J = (9, 8)$, $G = (4, 18)$, $C = (18, 0)$, $I = (11, 9)$, and $F = (14, 0)$. This can be done in a number of ways including by calculating half the pseudo-determinants

$$\frac{1}{2} \begin{vmatrix} 0 & 18 \\ 0 & 14 \\ 7 & 7 \\ 9 & 8 \\ 4 & 18 \\ 0 & 18 \end{vmatrix} = \frac{1}{2} \left[\begin{matrix} (0 \cdot 14 + 0 \cdot 7 + 7 \cdot 8 + 9 \cdot 18 + 4 \cdot 18) - \\ (0 \cdot 18 + 4 \cdot 8 + 9 \cdot 7 + 7 \cdot 14 + 0 \cdot 18) \end{matrix} \right] = \frac{97}{2} \quad \text{and} \quad \frac{1}{2} \begin{vmatrix} 18 & 0 \\ 11 & 9 \\ 7 & 7 \\ 14 & 0 \\ 18 & 0 \end{vmatrix} = \frac{1}{2} \left[\begin{matrix} (18 \cdot 9 + 11 \cdot 7 + 7 \cdot 0 + 14 \cdot 0) - \\ (18 \cdot 0 + 14 \cdot 7 + 7 \cdot 9 + 11 \cdot 0) \end{matrix} \right] = 39.$$

The desired ratio of areas is $\frac{\frac{97}{2}}{39} = \frac{97}{78}$, and the requested sum is $97 + 78 = 175$.

The Purple Comet! Math Meet staff apologizes and regrets that we all missed the typographical error in the original presentation of Problem 30. The problem originally asked for $4x^2 + 8y^3$ instead of the correct $8x^2 + 4y^3$. Since the original problem does not have a non-negative integer answer, this problem had to be eliminated from the competition.

Problem 30

Let x and y be real numbers satisfying

$$(x^2 + x - 1)(x^2 - x + 1) = 2(y^3 - 2\sqrt{5} - 1)$$

and

$$(y^2 + y - 1)(y^2 - y + 1) = 2(x^3 + 2\sqrt{5} - 1)$$

Find $8x^2 + 4y^3$.

Answer: 20

First note that the second equation is not the same as the first equation with the variables x and y interchanged. This shows that when the equations are satisfied, x and y are two distinct values. Rewrite the two equations as

$$x^4 - (x - 1)^2 = 2y^3 - 4\sqrt{5} - 2$$

$$y^4 - (y - 1)^2 = 2x^3 + 4\sqrt{5} - 2$$

and add the two equations to get $(x^4 - x^2 + 2x + 1 - 2x^3) + (y^4 - y^2 + 2y + 1 - 2y^3) = 0$ and $(x^2 - x - 1)^2 + (y^2 - y - 1)^2 = 0$. It follows that both $x^2 - x - 1 = 0$ and $y^2 - y - 1 = 0$ so x and y must be the two roots of $x^2 - x - 1$. The two values that satisfy the original equation are $x = \frac{1-\sqrt{5}}{2}$ and $y = \frac{1+\sqrt{5}}{2}$. Finally, $8x^2 + 4y^3 = 8\frac{1-2\sqrt{5}+(\sqrt{5})^2}{2^2} + 4\frac{1+3\sqrt{5}+3(\sqrt{5})^2+(\sqrt{5})^3}{2^3} = (12 - 4\sqrt{5}) + (8 + 4\sqrt{5}) = 20$.