PURPLE COMET MATH MEET April 2010

MIDDLE SCHOOL - SOLUTIONS

Problem 1

If $125 + n + 135 + 2n + 145 = 900$, find n.

Answer: 165

The given equation simplifies to $405 + 3n = 900$ which implies $3n = 495$, so $n = 165$.

Problem 2

Three boxes each contain four bags. Each bag contains five marbles. How many marbles are there altogether in the three boxes?

Answer: 60

There are $3 \cdot 4 = 12$ bags each containing 5 marbles, so there are $5 \cdot 12 = 60$ marbles.

Problem 3

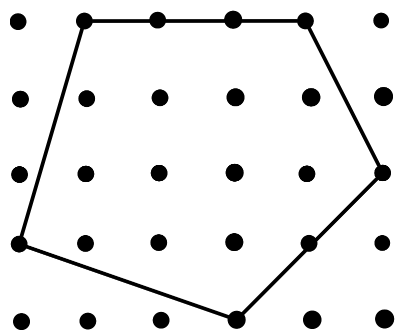
The sum $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = \frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

Answer: 69

Placing each fraction over a common denominator of 60, the sum becomes
$\frac{60}{60} + \frac{30}{60} + \frac{20}{60} + \frac{15}{60} + \frac{12}{60} + \frac{10}{60} = \frac{147}{60} = \frac{49}{20}$. The requested sum is $49 + 20 = 69$.

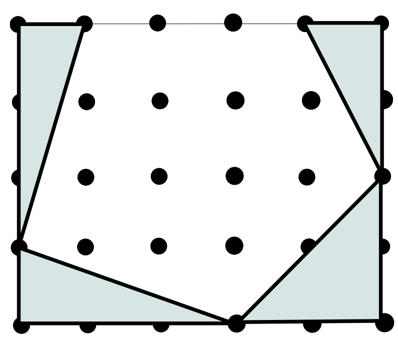
Problem 4

The grid below contains five rows with six points in each row. Points that are adjacent either horizontally or vertically are a distance one apart. Find the area of the pentagon shown.



Answer: 14

The pentagon is a four by five rectangle with four right triangles removed as shown. Two of the right triangles have legs of length 1 and 3. They each have area $\frac{1 \cdot 3}{2} = \frac{3}{2}$. The other two triangles have legs of length 1 and 2 and legs of length 2 and 2. These two triangles have areas $\frac{1 \cdot 2}{2} = 1$ and $\frac{2 \cdot 2}{2} = 2$, respectively. Thus, the pentagon has area $4 \cdot 5 - \frac{3}{2} - \frac{3}{2} - 1 - 2 = 14$.



Problem 5

Find the least positive integer k so that $k + 25973$ is a palindrome (a number which reads the same forward and backwards).

Answer: 89

The five digit palindrome beginning 259 is 25952 which does not exceed 25973, so the least palindrome exceeding 25973 must begin 260 and is 26062. The desired $k = 26062 - 25973 = 89$.

Problem 6

Find the sum of the prime factors of 777.

Answer: 47

The number 777 factors as $3 \cdot 7 \cdot 37$, so the requested sum is $3 + 7 + 37 = 47$.

Problem 7

x and y are positive real numbers where x is p percent of y, and y is $4p$ percent of x. What is p?

Answer: 50

The problem states that $x = \frac{p}{100}y$ and $y = \frac{4p}{100}x$. Replacing x in the second equation with its value from the first equation yields $y = \frac{4p}{100} \cdot \frac{p}{100}y = \frac{4p^2}{100^2}y$. Since y is not zero, this reduces to $p^2 = \frac{100^2}{4}$, so $p = \frac{100}{2} = 50$.

Problem 8

There are exactly two four-digit numbers that are multiples of three where their first digit is double their second digit, their third digit is three more than their fourth digit, and their second digit is 2 less than their fourth digit. Find the difference of these two numbers.

Answer: 6333

If such a number has a second digit of m, then its first digit is $2m$, its fourth digit is $m + 2$, and its third digit is $m + 5$. The sum of the four digits is then $m + 2m + (m + 5) + (m + 2) = 5m + 7$. To be a multiple of 3, the digits must add to a multiple of 3, so $5m + 7$ must be a multiple of 3. This happens when m is 1, 4, or 7, but when m is 7, $m + 5$ is not a digit. Thus, the two numbers are 2163 (when $m = 1$) and 8496 (when $m = 4$). The difference of these two numbers is 8496 - 2163 = 6333.

Problem 9

What percent of the numbers 1, 2, 3, ... 1000 are divisible by exactly one of the numbers 4 and 5?

Answer: 35

Exactly one quarter of the numbers 1, 2, 3, ... 1000 are divisible by 4 which is $\frac{1000}{4} = 250$ numbers. Exactly one fifth of the numbers are divisible by 5 which is $\frac{1000}{5} = 200$ numbers. Some of the numbers are divisible by both 4 and 5. They are the ones divisible by 20, so there are exactly $\frac{1000}{20} = 50$ numbers divisible by both 4 and 5. Thus, there are 250 - 50 numbers divisible by 4 and not divisible by 5, and there are 200 - 50 = 150 numbers divisible by 5 and not divisible by 4. This shows there are 200 + 150 = 350 numbers divisible by exactly one of 4 and 5 which is $\frac{350}{1000} \cdot 100 = 35$ percent of the numbers.

Problem 10

A baker uses $6\frac{2}{3}$ cups of flour when she prepares $\frac{5}{3}$ recipes of rolls. She will use $9\frac{3}{4}$ cups of flour when she prepares $\frac{m}{n}$ recipes of rolls where m and n are relatively prime positive integers. Find $m + n$.

Answer: 55

Since the ratio of the number of recipes made to the cups of flour used must be constant, $\frac{\frac{5}{3}}{6\frac{2}{3}} = \frac{\frac{m}{n}}{9\frac{3}{4}}$. Thus, $\frac{\frac{5}{3}}{\frac{20}{3}} = \frac{\frac{m}{n}}{\frac{39}{4}}$ and $\frac{m}{n} = \frac{\frac{5}{3}}{\frac{20}{3}} \frac{39}{4} = \frac{39}{16}$. The requested sum is $39 + 16 = 55$.

Problem 11

There are two rows of seats with three side-by-side seats in each row. Two little boys, two little girls, and two adults sit in the six seats so that neither little boy sits to the side of either little girl. In how many different ways can these six people be seated?

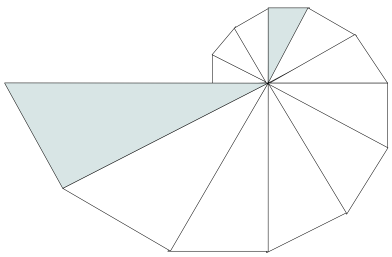
Answer: 176

First observe that the two adults may not sit in the same row. Since there are three seats in each row, there are $3 \cdot 3 = 9$ ways of selecting two seats for the adults to sit. If the two adults sit in the two center seats in each row, then the four children can sit in any of the other four seats. There are 2 ways for the adults to seat themselves in the two central seats, and there are $4! = 24$ ways for the children to sit in the four other seats. Thus, there are $2 \cdot 24 = 48$ ways for the people to sit with the adults in the two center seats.

If either adult sits in an end seat, then there are two seats next to that adult where either two little boys or two little girls will need to sit. Thus, the little boys will need to sit in one row, and the little girls will need to sit in the other row. There are $9 - 1 = 8$ ways to choose the pair of seats for the adults to sit so that at least one adult sits in an end seat. There are two ways for the adults to choose which of the two seats to sit in, so there are $2 \cdot 8 = 16$ ways for the adults to choose seats. There are two ways for the little boys to choose a row to sit in. Then there are $2 \cdot 2$ ways for the little boys and little girls to choose seats. This accounts for $16 \cdot 2 \cdot 2 \cdot 2 = 128$ ways for the six people to sit. Thus, the final count is $48 + 128 = 176$.

Problem 12

The diagram below shows twelve 30-60-90 triangles placed in a circle so that the hypotenuse of each triangle coincides with the longer leg of the next triangle. The fourth and last triangle in this diagram are shaded. The ratio of the perimeters of these two triangles can be written as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.



Answer: 337

In a 30-60-90 triangle, the ratio of the length of the longer leg to the length of the hypotenuse is $\frac{\sqrt{3}}{2}$. So, since all twelve triangles are similar, the ratio of the length of the hypotenuse of one of the triangles in the diagram to the length of the hypotenuse of the next larger triangle is the same $\frac{\sqrt{3}}{2}$. It follows that the perimeters of two adjacent triangles are in this same ratio. Thus, the ratio of the perimeter of the fourth triangle to the perimeter of the twelfth triangle is $\left(\frac{\sqrt{3}}{2}\right)^{12-4} = \frac{3^4}{2^8} = \frac{81}{256}$. The requested sum is $81 + 256 = 337$.

Problem 13

Find the number of sets A that satisfy the three conditions:

- A is a set of two positive integers
- each of the numbers in A is at least 22 percent the size of the other number
- A contains the number 30.

Answer: 129

The set A contains the number 30. The smallest integer which could be a member of A is $7 > 30 \cdot 0.22 = 6.6$. The greatest integer that could be in A is $136 < 30 \div 0.22 = 136.\overline{36}$. So the second number in A can be any integer between 7 and 136 except for the number 30. Thus, the number of possible sets A is $136 - 6 - 1 = 129$.

Problem 14

Let $ABCD$ be a trapezoid where $\overline{AB}$ is parallel to $\overline{CD}$. Let P be the intersection of diagonal $\overline{AC}$ and diagonal $\overline{BD}$. If the area of triangle PAB is 16, and the area of triangle PCD is 25, find the area of the trapezoid.

Answer: 81

Let $CD = x$, and y be the length of the altitude of triangle PCD from P to $\overline{CD}$. Then $\frac{xy}{2} = 25$. Let α be the ratio of AB to CD. Note that triangles PAB and PCD are similar, so the area of triangle PAB must be $25\alpha^2 = 16$, and $\alpha = \frac{4}{5}$. The trapezoid has bases of lengths x and αx and a height of $y + \alpha y$. It follows that the trapezoid has area $\frac{x+\alpha x}{2}(y + \alpha y) = \frac{xy}{2}(1+\alpha)^2 = 25\left(1 + \frac{4}{5}\right)^2 = 25\left(\frac{9}{5}\right)^2 = 81$.

Problem 15

In the number arrangement

```
1
2    3
4    5    6
7    8    9    10
11   12   13   14   15
.
.
.
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what is the number that will appear directly below the number 2010?

Answer: 2073

The final number in row k is the k^{th} triangular number $\frac{k(k+1)}{2}$. A little investigating shows that row 62 ends with the number $\frac{62\cdot 63}{2} = 1953$, and row 63 ends with the number $\frac{63\cdot 64}{2} = 2016$. It follows that 2010 is number $2010 - 1953 = 57$ in row 62. The number directly below 2010 is number 57 in row 63 which is $2016 + 57 = 2073$.

Problem 16

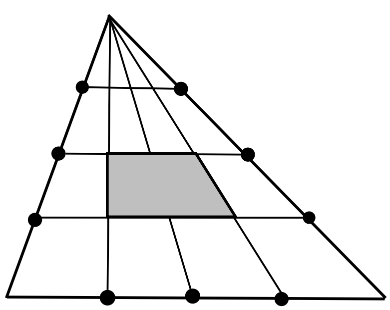
Half the volume of a 12 foot high cone-shaped pile is grade A ore while the other half is grade B ore. The pile is worth \$62. One-third of the volume of a similarly shaped 18 foot pile is grade A ore while the other two-thirds is grade B ore. The second pile is worth \$162. Two-thirds of the volume of a similarly shaped 24 foot pile is grade A ore while the other one-third is grade B ore. What is the value in dollars (\$) of the 24 foot pile?

Answer: 608

Let a 12 foot pile of grade A ore be worth x and a 12 foot pile of grade B ore be worth y. Then the problem states that $\frac{1}{2}x + \frac{1}{2}y = 62$ which simplifies to $x + y = 124$. Since the 18 foot pile has a volume $\left(\frac{18}{12}\right)^3 = \frac{27}{8}$ times that of the 12 foot pile, the problem states that $\frac{27}{8}\left(\frac{1}{3}x + \frac{2}{3}y\right) = 162$ which simplifies to $x + 2y = 144$. The 24 foot pile then has worth $8\left(\frac{2}{3}x + \frac{1}{3}y\right) = \frac{8}{3}\left(2x + y\right) = \frac{8}{3}\left(3(x+y) - (x+2y)\right) = \frac{8}{3}\left(3 \cdot 124 - 144\right) = 608$. Note that $x = 104$ and $y = 20$, but these values are not needed to find the solution.

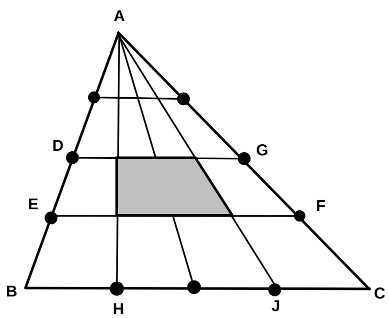
Problem 17

The diagram below shows a triangle divided into sections by three horizontal lines which divide the altitude of the triangle into four equal parts, and three lines connecting the top vertex with points that divide the opposite side into four equal parts. If the shaded region has area 100, find the area of the entire triangle.



Answer: 640

Label the vertices of the triangle and intersection points as shown. Let x be the area of triangle ABC.



Triangle AEF is similar to triangle ABC with an altitude that is $\frac{3}{4}$ the length, so the area of triangle AEF is $x\left(\frac{3}{4}\right)^2$. Triangle ADG is similar to triangle ABC with an altitude that is $\frac{1}{2}$ the length, so the area of triangle ADG is $x\left(\frac{1}{2}\right)^2$. Finally, the shaded region has half the area of trapezoid $DEFG$ because both bases are half as long. Thus, $100 = \frac{1}{2}\left[x\left(\frac{3}{4}\right)^2 - x\left(\frac{1}{2}\right)^2\right)$ and $100 = x \cdot \frac{1}{2} \cdot \left(\frac{9}{16} - \frac{1}{4}\right)$. This gives $x = 100 \cdot 2 \cdot \frac{16}{5} = 640$.

Problem 18

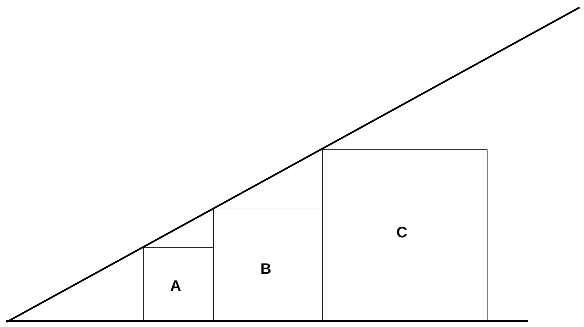
How many three-digit positive integers contain both even and odd digits?

Answer: 675

There are $999 - 99 = 900$ three digit positive integers. Since there are 5 odd digits, $5^3 = 125$ of the three-digit positive integers contain only odd digits. Since the first digit of a three-digit integer cannot be zero, there are $4 \cdot 5^2 = 100$ three-digit positive integers that contain only even digits. Thus, there are $900 - 125 - 100 = 675$ three-digit positive integers that contain both even and odd digits.

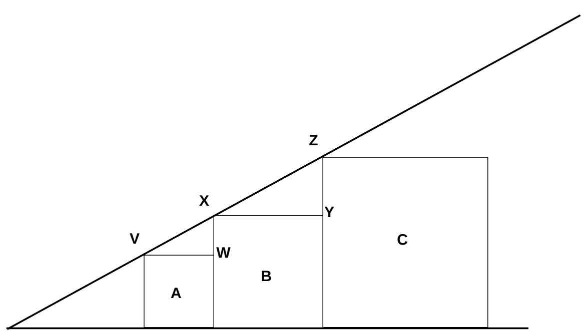
Problem 19

Square A is adjacent to square B which is adjacent to square C. The three squares all have their bottom sides along a common horizontal line as shown. The upper left vertices of the three squares are collinear. If square A has area 24, and square B has area 36, find the area of square C.



Answer: 54

Let the upper corners of square A be labeled V and W, the upper corners of square B be labeled X and Y, and the upper left corner of square C be labeled Z as shown below. Since the sides of the squares are parallel, the triangles VWX and XYZ are similar. Note that since square A has area 24, it has side length $VW = \sqrt{24} = 2\sqrt{6}$, and since square B has area 36, it has side length $XY = \sqrt{36} = 6$. The ratio of corresponding sides of similar triangles are equal, so $\frac{WX}{VW} = \frac{YZ}{XY}$ and $YZ = XY \cdot \frac{WX}{VW} =$ $6 \cdot \frac{6-2\sqrt{6}}{2\sqrt{6}} = 3\sqrt{6} - 6$. The side length of square C is ZY plus the length of the side of square B, so the side length of square C is $(3\sqrt{6} - 6) + 6 = 3\sqrt{6}$. The area of square C is $\left(3\sqrt{6}\right)^2 = 9 \cdot 6 = 54$.



Problem 20

Suppose that f is a function such that $3f(x) - 5xf(\frac{1}{x}) = x - 7$ for all non-zero real numbers x. Find $f(2010)$.

Answer: 4021

If x is not zero, replace x by $\frac{1}{x}$ in the equation to obtain $3f(\frac{1}{x}) - 5\frac{1}{x}f(x) = \frac{1}{x} - 7$. By multiplying by x, this reduces to $3xf(\frac{1}{x}) - 5f(x) = 1 - 7x$. Adding 5 times this equation to 3 times the original equation yields

$$5\left[3xf(\frac{1}{x}) - 5f(x)\right] + 3\left[3f(x) - 5xf(\frac{1}{x})\right] = 5[1 - 7x] + 3[x - 7]$$

This simplifies to $-16f(x) = -16 - 32$ and $f(x) = 1 + 2x$. It follows that $f(2010) = 1 + 2(2010) = 4021$.