PURPLE COMET MATH MEET April 2009

HIGH SCHOOL - SOLUTIONS

Problem 1

In five years, Tom will be twice as old as Cindy. Thirteen years ago, Tom was three times as old as Cindy. How many years ago was Tom four times as old as Cindy?

Answer: 19

Let Tom be age t, and Cindy be age c. Then the first two statements of the problem say $t + 5 = 2(c + 5)$ and $t - 13 = 3(c - 13)$. The first of these two equations simplifies to $t = 2c + 5$. Substituting this into the second equation gives $(2c + 5) - 13 = 3(c - 13)$ which simplifies to $2c - 8 = 3c - 39$ and $c = 31$. So, $t = 2c + 5 = 2(31) + 5 = 67$. Cindy is now 31, and Tom is 67. The answer to the problem is n where $67 - n = 4(31 - n)$. This simplifies to $67 - n = 124 - 4n$ and $3n = 57$. Thus, $n = 19$, and Tom was four times as old as Cindy 19 years ago.

Problem 2

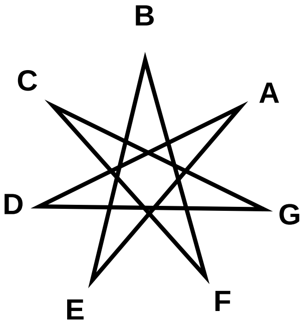
Find the least positive integer n such that for every prime number p, $p^2 + n$ is never prime.

Answer: 5

$2^2 + 1 = 5$, $3^2 + 2 = 11$, $2^2 + 3 = 7$, and $3^2 + 4 = 13$ are all prime numbers showing that n cannot be 1, 2, 3, or 4. When p is 2, $p^2 + 5 = 9$ which is not prime. When p is a prime greater than 2, $p^2 + 5$ is an even number greater than 2, so it is not prime. Thus, 5 is the least integer satisfying the given condition.

Problem 3

In the diagram $ABCDEFG$ is a regular heptagon (a 7 sided polygon). Shown is the star $AEBFCGD$. The degree measure of the obtuse angle formed by AE and CG is $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m+n$.



Answer: 727

Let H be the point at the intersection of segments AE and CG. The desired angle AHC is an exterior angle of triangle AHG, so the measure of the desired angle is equal to the sum of the measures of angle HGA and angle HAG. These two angles are congruent. Angle GAB is an angle of a regular heptagon, so its measure is $\frac{180\cdot(7-2)}{7}$. The vertices of the regular heptagon lie on a circle, and the arc of that circle from point D to point E measures $\frac{360}{7}$, and, thus, angle DAE measures $\frac{180}{7}$. Therefore, the desired angle AHC is twice the angle HAG, and this is the difference between the measures of angle BAG and angle DAE. The desired measure is $\frac{180\cdot 5}{7} - \frac{180}{7} = \frac{720}{7}$. The requested sum is $720 + 7 = 727$.

Alternatively, one can rotate the line AE to line BE, then rotate line BE to line BF, then rotate line BF to line CF, and finally rotate line CF to line CG. Each of these four rotations is, as seen above, a rotation through an angle of $\frac{180}{7}$. The four rotations end up rotating line AE through an angle of $4\cdot\frac{180}{7} = \frac{720}{7}$.

Problem 4

There are three bags of marbles. Bag two has twice as many marbles as bag one. Bag three has three times as many marbles as bag one. Half the marbles in bag one, one third the marbles in bag two, and one fourth the marbles in bag three are green. If all three bags of marbles are dumped into a single pile, $\frac{m}{n}$ of the marbles in the pile would be green where m and n are relatively prime positive integers. Find $m+n$.

Answer: 95

Assume that the first bag contains K marbles. Then the second bag contains $2K$ marbles, and the third bag contains $3K$ marbles. The total number of green marbles is $\frac{1}{2}\cdot K+\frac{1}{3}\cdot 2K+\frac{1}{4}\cdot 3K$, and the total number of marbles is $K+2K+3K=6K$. The desired fraction is therefore $\frac{\frac{1}{2}\cdot K+\frac{1}{3}\cdot 2K+\frac{1}{4}\cdot 3K}{6K}=\frac{\frac{1}{2}+\frac{2}{3}+\frac{3}{4}}{6}=\frac{23}{72}$. The requested sum is 23 + 72 = 95.

Problem 5

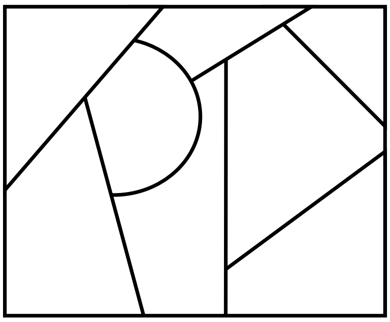
Find n so that $\left(4^{n+7}\right)^3=\left(2^{n+23}\right)^4$.

Answer: 25

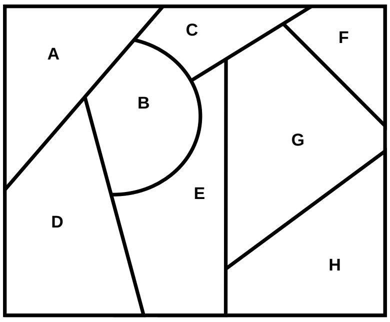
Simplify the equation $\left(4^{n+7}\right)^3=\left(2^{n+23}\right)^4$ to get $\left(2^2\right)^{3(n+7)}=2^{4(n+23)}$ and $2^{6(n+7)}=2^{4(n+23)}$. It follows that $6n+42=4n+92$ and $2n=50$. Thus, $n=25$.

Problem 6

Wiles county contains eight townships as shown on the map. If there are four colors available, in how many ways can the the map be colored so that each township is colored with one color and no two townships that share a border are colored with the same color?



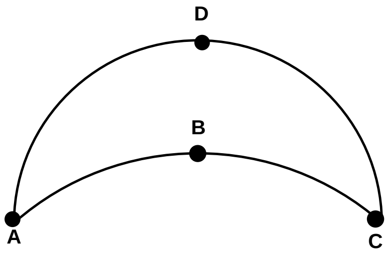
Answer: 576



Label the townships as shown. There are four choices for colors for township A, and three choices for colors for township B. If townships C and D have the same color, then there are two choices for the color of C and D, and there are two choices for colors for each of the remaining townships E, F, G, and H. If townships C and D have different colors, then there are two choices for the color of C and one choice for a color for D. Since B, C, and D will all have different colors, there is only one choice of color for township E. Then there are two choices for colors for each of the remaining townships F, G, and H. The total number of ways to color the eight townships is then $4 \cdot 3 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 + 4 \cdot 3 \cdot 2 \cdot 1 \cdot 2 \cdot 2 \cdot 2 = 4 \cdot 3 \cdot 2 \cdot 3 \cdot 2 \cdot 2 \cdot 2 = 576$.

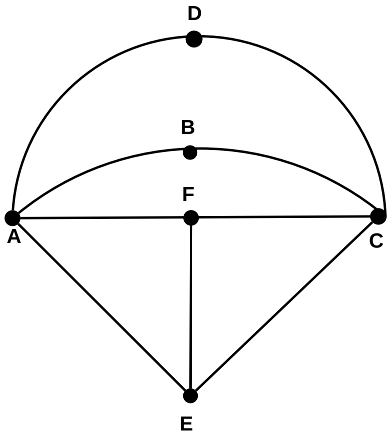
Problem 7

The figure $ABCD$ is bounded by a semicircle CDA and a quarter circle ABC. Given that the distance from A to C is 18, find the area of the figure.



Answer: 81

Let F be the center of the semicircle, and E be the center of the quarter circle as shown. The requested area is the area of semicircle $AFCD$ minus the area of quarter circle $AECB$ plus the area of the triangle AEC. Since $|FC|$ is 9 and $|EC|$ is $9\sqrt{2}$, the requested area is $\frac{9^2\pi}{2} - \frac{(9\sqrt{2})^2\pi}{4} + \frac{18 \cdot 9}{2} = 81$.



Problem 8

Find the least positive integer that has exactly 20 positive integer divisors.

Answer: 240

If p is a prime number, then the prime power p^a has $a+1$ divisors. A product of prime powers $p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k}$ has $(a_1+1)(a_2+1)\cdots(a_k+1)$ divisors. For a number to have 20 positive integer divisors, $(a_1+1)(a_2+1)\cdots(a_k+1)$ must be $20 = 2\cdot 10 = 4\cdot 5 = 2\cdot 2\cdot 5$. The minimal positive integers corresponding to these four possibilities are $2^{19} = 524,288$, $2^9\cdot 3 = 1536$, $2^4\cdot 3^3 = 432$, and $2^4\cdot 3\cdot 5 = 240$. The least of these is 240.

Problem 9

Bill bought 13 notebooks, 26 pens, and 19 markers for 25 dollars. Paula bought 27 notebooks, 18 pens, and 31 markers for 31 dollars. How many dollars would it cost Greg to buy 24 notebooks, 120 pens, and 52 markers?

Answer: 88

If notebooks cost x dollars, pens cost y dollars, and markers cost z dollars, then the problem states that $13x+26y+19z = 25$ and $27x+18y+31z = 31$. It follows that the cost in dollars to Greg would be $24x+120y+52z = 6(13x+26y+19z)-2(27x+18y+31z) = 6(25)-2(31) = 88$.

Problem 10

Towers grow at points along a line. All towers start with height 0 and grow at the rate of one meter per second. As soon as any two adjacent towers are each at least 1 meter tall, a new tower begins to grow at a point along the line exactly half way between those two adjacent towers. Before time 0 there are no towers, but at time 0 the first two towers begin to grow at two points along the line. Find the total of all the heights of all towers at time 10 seconds.

Answer: 1033

Two towers begin to grow at time 0. At time 1, the first two towers reach a height of 1 meter, so one new tower begins to grow. At time 2, the new tower reaches a height of 1, so two new towers begin to grow, one between the new tower and the first old tower, and one between the new tower and the second old tower. Each second thereafter, twice as many new towers begin to grow, so at time 3, 2^2 towers begin, at time 4, 2^3 towers begin, and so forth until at time 9, 2^8 towers begin to grow. A tower that begins to grow at time k reaches a height of $10-k$ by time 10. Thus, the total height of all towers at time 10 is $2\cdot 10+\sum_{k=1}^{9}2^{k-1}(10-k) = 20+\sum_{k=1}^{9}\sum_{j=1}^{k}2^{j-1} = 20+\sum_{k=1}^{9}(2^k-1) = 20+2^{10}-11 = 9+1024 = 1033$.

Problem 11

The four points $A(-1,2)$, $B(3,-4)$, $C(5,-6)$, and $D(-2,8)$ lie in the coordinate plane. Compute the minimum possible value of $PA+PB+PC+PD$ over all points P.

Answer: 23

Let P_0 be the intersection of AC and BD. Note that P_0 is between A and C, and also between B and D. Therefore, if P is not P_0, then by the triangle inequality either $PA+PC > P_0A+P_0C$ or $PB+PD > P_0B+P_0D$, or both. Thus, the quantity has its minimum value at P_0, and this quantity is equal to $AC+BD$.

$AC = \sqrt{[5-(-1)]^2+[-6-2]^2} = 10$

$BD = \sqrt{[-2-3]^2+[8-(-4)]^2} = 13$

So the minimum value of the desired quantity is $10+13=23$.

Problem 12

What is the least possible sum of two positive integers a and b where $a \cdot b = 10!$?

Answer: 3810

$10! = 1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9\cdot10 = 2^8\cdot3^4\cdot5^2\cdot7$. To make the sum $a+b$ as small as possible, a and b should be as close to each other as possible where $a \cdot b$ contains 8 factors of 2, 4 factors of 3, 2 factors of 5, and a factor of 7. This can be found by finding the best way to approximate 7 as even powers of 2, 3, and 5. For example, 7 is close to 3^2, closer to $\frac{5^2}{2^2}$, but very close to $\frac{2^6}{3^2}$. Thus, the optimal factoring of $10!$ is found by taking $2\cdot3\cdot5$ and multiplying it by either 2^6 or $3^2\cdot7$ to get the factoring $(2^7\cdot3\cdot5)(2\cdot3^3\cdot5\cdot7) = 1920\cdot1890$. Thus, the least sum is $1920+1890=3810$.

Problem 13

Greta is completing an art project. She has twelve sheets of paper: four red, four white, and four blue. She also has twelve paper stars: four red, four white, and four blue. She randomly places one star on each sheet of paper. The probability that no star will be placed on a sheet of paper that is the same color as the star is $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $n-100m$.

Answer: 25

$\frac{m}{n}$ is the probability that if we separate the twelve stars into a red set, a white set, and a blue set, all size four, no star gets placed into the set of its color. The number of equally likely ways of placing 12 stars into 3 distinguishable sets of 4 is given by the multinomial coefficient $\binom{12}{4,4,4} = \frac{12!}{4!\cdot4!\cdot4!}$. Now count the number of ways of placing the stars so that no star ends up in a set with the same color as the star. In such a placement, let r be the number of red stars in the white set. Then there are $4 - r$ blue stars in the white set, so there will be r blue stars in the red set. Then there are $4 - r$ white stars in the red set, so there will be r white stars in the blue set. It follows that the number of placements with r red stars in the white set is $\binom{4}{r}^3$. Thus, the number of placements of stars so that no star ends up in a set with the same color as the star is $\sum_{r=0}^{4} \binom{4}{r}^3 = 1^3 + 4^3 + 6^3 + 4^3 + 1^3 = 346$. Because $\binom{12}{4,4,4} = \frac{12\cdot11\cdot10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{4\cdot3\cdot2\cdot1\cdot4\cdot3\cdot2\cdot1\cdot4\cdot3\cdot2\cdot1} = 34,650$, the probability is $\frac{364}{36450} = \frac{173}{17325}$, and the requested difference is $17325 - 100 \cdot 173 = 25$.

Problem 14

Let $ABCD$ be a trapezoid with AB parallel to CD, AB has length 1, and CD has length 41. Let points X and Y lie on sides AD and BC, respectively, such that XY is parallel to AB and CD, and XY has length 31. Let m and n be two relatively prime positive integers such that the ratio of the area of $ABYX$ to the area of $CDXY$ is $\frac{m}{n}$. Find $m + 2n$.

Answer: 10

Let E be the intersection of line AD and line BC. Then triangles EAB, EXY, and EDC are similar. If triangle EAB has area α, then EXY has area $31^2\alpha$, and EDC has area $41^2\alpha$. It follows that the desired ratio is $\frac{31^2\alpha-\alpha}{41^2\alpha-31^2\alpha} = \frac{961-1}{1681-961} = \frac{960}{720} = \frac{4}{3}$, and the requested sum is $4 + 2 \cdot 3 = 10$.

Problem 15

What is the remainder when 7^{8^9} is divided by 1000?

Answer: 801

The Euler ϕ function tells how many positive integers less than a given integer are relatively prime to that integer. Thus, $\phi(1000) = \phi(8) \cdot \phi(125) = 4 \cdot 100 = 400$. Since for positive integers a and n the power $a^{\phi(n)}$ is congruent to 1 mod n, it follows that $7^{400} = 1 \mod 1000$. So, $7^{8^9} \mod 1000 = 7^{8^9 \mod 400}$. Here $8^9 \equiv 2^{27} \equiv 2^{11}2^{11}2^5 \equiv 2048 \cdot 2048 \cdot 32 \equiv 48^2 \cdot 32 \equiv 96^2 \cdot 8 \equiv (100-4)^2 \cdot 8 \equiv 16 \cdot 8 \equiv 128 \mod 400$. Then $7^{8^9} \equiv 7^{128} \mod 1000$. Note that $7^2 = 49$ so

$$7^4 = (50-1)^2 = 2500 - 2 \cdot 50 + 1 = 2401 \equiv 401 \mod 1000$$

$$7^8 = (400+1)^2 = 160,000 + 800 + 1 \equiv 801 \mod 1000$$

$$7^{16} = (800+1)^2 = 640,000 + 1600 + 1 \equiv 601 \mod 1000$$

$$7^{32} = (600+1)^2 = 360,000 + 1200 + 1 \equiv 201 \mod 1000$$

$$7^{64} = (200+1)^2 = 40,000 + 400 + 1 \equiv 401 \mod 1000$$

It follows that $7^{8^9} \equiv 801 \mod 1000$.

Problem 16

Let the complex number $z = \cos\frac{1}{1000} + i\sin\frac{1}{1000}$. Find the smallest positive integer n so that z^n has an imaginary part which exceeds $\frac{1}{2}$.

Answer: 524

By DeMoivre's Theorem, $z^n = \cos\frac{n}{1000} + i\sin\frac{n}{1000}$. So n must satisfy $\sin\frac{n}{1000} > \frac{1}{2}$, and $\frac{n}{1000} > \frac{\pi}{6}$. Thus, $n > \frac{1000\pi}{6} > \frac{3141.59}{6} > 523$. Therefore, the smallest n satisfying the condition is $n = 524$.

Problem 17

How many ordered triples (a, b, c) of odd positive integers satisfy $a + b + c = 25$?

Answer: 78

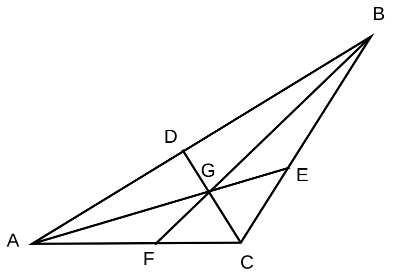
Let (a, b, c) be an ordered triple of odd positive integers satisfying $a + b + c = 25$. Then $(a-1) + (b-1) + (c-1) = 22$ and $(x, y, z) = \left(\frac{a-1}{2}, \frac{b-1}{2}, \frac{c-1}{2}\right)$ is an ordered triple of nonnegative integers satisfying $x + y + z = 11$. Since there is a one-to-one correspondence between the ordered triples (a, b, c) in the problem and the ordered triples of nonnegative integers (x, y, z) satisfying $x + y + z = 11$, the answer is the number of ways to place 11 indistinguishable items into three distinguishable containers. This problem can be solved using the well-known *stars and bars* technique of counting the arrangements of 11 stars and two separating bars or $\binom{11+2}{2} = \binom{13}{2} = \frac{13\cdot 12}{2} = 78$.

Alternatively, a solution to $a + b + c = 25$ in non-negative integers corresponds to any linear arrangement of 25 stars and 2 bars. One can obtain such an arrangement by starting with a list of 27 starts and changing two of the stars into bars. For the arrangement to correspond to a solution involving only odd integers, the two stars changed to bars must be stars in even numbered positions. There are 13 even numbered positions. The number of ways of choosing two such positions is, as above, $\binom{13}{2} = \frac{13\cdot 12}{2} = 78$.

Problem 18

On triangle ABC let D be the point on AB so that CD is an altitude of the triangle, and E be the point on BC so that AE bisects angle BAC. Let G be the intersection of AE and CD, and let point F be the intersection of side AC and the ray BG. If AB has length 28, AC has length 14, and CD has length 10, then the length of CF can be written as $\frac{k-m\sqrt{p}}{n}$ where k, m, n, and p are positive integers, k and n are relatively prime, and p is not divisible by the square of any prime. Find $k - m + n + p$.

Answer: 623



By the Pythagorean theorem AD has length $\sqrt{14^2 - 10^2} = 4\sqrt{6}$. Then BD has length $28 - 4\sqrt{6}$. By the angle bisector theorem, $\frac{BE}{CE} = \frac{AB}{AC} = 2$. It then follows from Ceva's Theorem that
$1 = \frac{AD}{BD} \cdot \frac{BE}{CE} \cdot \frac{CF}{AF} = \frac{4\sqrt{6}}{28-4\sqrt{6}} \cdot 2 \cdot \frac{CF}{14-CF}$. Thus, $14 - CF = \frac{2\sqrt{6}}{7-\sqrt{6}} \cdot CF$ or
$CF = 14 \cdot \frac{7-\sqrt{6}}{7+\sqrt{6}} = 14 \cdot \frac{7^2-14\sqrt{6}+6}{7^2-6} = \frac{770-196\sqrt{6}}{43}$. The requested sum is then $770 - 196 + 43 + 6 = 623$.

Problem 19

If a and b are complex numbers such that $a^2 + b^2 = 5$ and $a^3 + b^3 = 7$, then their sum, $a + b$, is real. The greatest possible value for the sum $a + b$ is $\frac{m+\sqrt{n}}{2}$ where m and n are integers. Find n.

Answer: 57

Let $s = a + b$ and $r = a \cdot b$. Then $s^2 - 2r = (a + b)^2 - 2a \cdot b = a^2 + b^2 = 5$ and
$s^3 - 3r \cdot s = (a + b)^3 - 3a \cdot b \cdot (a + b) = a^3 + b^3 = 7$. It follows that
$14 = 2s^3 - 3(2r)s = 2s^3 - 3(s^2 - 5)s = 2s^3 - 3s^3 + 15s$. Then $s^3 - 15s + 14 = (s - 1)(s^2 + s - 14) = 0$.
Therefore, s can be 1 or $\frac{-1\pm\sqrt{1+4\cdot 14}}{2} = \frac{-1\pm\sqrt{57}}{2}$. The greatest possible value for $a + b = s$ is $\frac{-1+\sqrt{57}}{2}$, and the requested answer is 57.

Problem 20

Five men and seven women stand in a line in random order. Let m and n be relatively prime positive integers so that $\frac{m}{n}$ is the probability that each man stands next to at least one woman. Find $m + n$.

Answer: 287

Twelve people stand in a line. There are $\binom{12}{5} = 792$ equally likely ways to select the five positions where men will stand. To find the requested probability, count the number of these arrangements where each man stands next to at least one woman. Clearly, in such an arrangement no three men can stand together in the line, and no two men can stand together at either end of the line. The five men can all stand separately forming groups arranged as 1-1-1-1-1, or two can stand together forming groups arranged as 2-1-1-1, 1-2-1-1, 1-1-2-1, or 1-1-1-2, or two groups of men can stand together forming groups arranged as 2-2-1, 2-1-2, or 1-2-2.

For men to stand in groups 1-1-1-1-1, four women must stand between these men, and the other three women can stand in any of the six positions to the left of, to the right of, or between the men. The number of such arrangements can be counted using the well known stars and bars technique with three stars (women) and five bars (men) to give $\binom{8}{3} = 56$ arrangements where no two men stand together.

For men to stand in groups 2-1-1-1, three women must stand between the groups of men and one woman must stand to the left of the group of two men. The remaining three women can stand in any of five positions. Thus, there are $\binom{7}{3} = 35$ such arrangements. There are also 35 arrangements with the groupings 1-1-1-2. For men to stand in groups 1-2-1-1, three women must stand between the groups of men, and the remaining four women can stand in any of five positions. Thus, there are $\binom{8}{4} = 70$ such arrangements. There are also 70 arrangements with the groupings 1-1-2-1. This accounts for $35 + 35 + 70 + 70 = 210$ arrangements with one group of two men standing together.

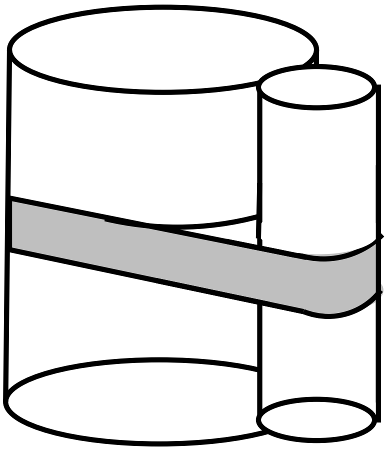
For men to stand in groups 2-2-1, two women must stand between the groups of men and one woman must stand to the left of all the men. The remaining four women can stand in any of four positions. Thus, thee are $\binom{7}{3} = 35$ such arrangements. There are also 35 arrangements the groupings 1-2-2. For men to stand in groups 2-1-2, two women are needed to stand between the groups of men, and two women are needed to stand at the far right and far left of the men. The remaining three women can stand in any of four positions. Thus, there are $\binom{6}{3} = 20$ arrangements with the men grouped 2-1-2. This accounts for $35 + 35 + 20 = 90$ arrangements with two groups of men standing together.

The total number of arrangements of positions where men can stand so that every man stands next to at least one woman is $56 + 210 + 90 = 356$, and the required probability is $\frac{356}{792} = \frac{89}{198}$. The requested sum is $89 + 198 = 287$.

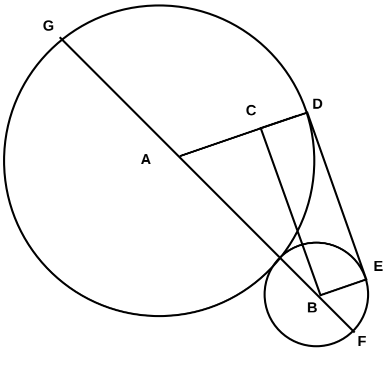
Alternatively, one can count the number of ways the men can stand so that at least one man does not stand next to a woman. If two men stand on the left end of the line, the other three men can stand in any of 10 positions in $\binom{10}{3} = 120$ ways. Similarly, there are 120 ways for two men to stand on the right end of the line, but 8 of these have two men at each end of the line. The total number of ways for two men to stand at the end of the line is $120 + 120 - 8 = 232$. For exactly three men to stand together, not at the end of the line, the three men can be placed in any of the six spaces between the women, and the other two men can stand individually in any of seven other positions or stand together in any of five positions. So the number of ways exactly three men can stand together is $6\left(\binom{7}{2} + 5\right) = 6(21 + 5) = 156$. Exactly four men can stand together in one of the six spaces between the women, and the fifth man can stand in one of seven other spaces for a total of $6 \cdot 7 = 42$ ways. Finally, there are six ways for all five men to stand together. This gives a total of $232 + 156 + 42 + 6 = 436$. This leaves $792 - 436 = 356$ ways for the men to stand so that each man stands next to at least one woman. As above, this gives the answer 287.

Problem 21

A cylinder radius 12 and a cylinder radius 36 are held tangent to each other with a tight band. The length of the band is $m\sqrt{k} + n\pi$ where m, k, and n are positive integers, and k is not divisible by the square of any prime. Find $m + k + n$.



Answer: 107

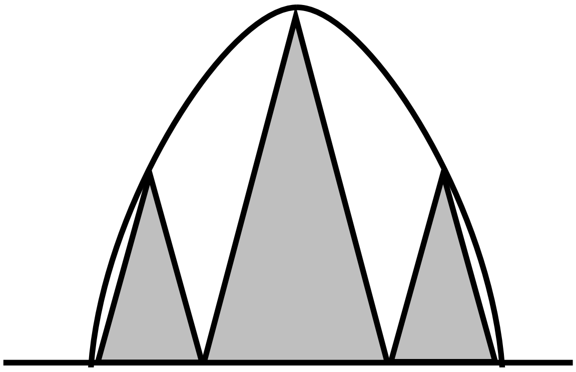


The diagram shows a cross section of the two cylinders and the band. Here the line segment FG passes through A and B, the centers of the large cylinder and small cylinder, respectively. Points D and E are where the band is tangent to the large cylinder and small cylinder, respectively. Then DE is perpendicular to radii AD and BE. Let C be the point on AD so that CB is parallel to DE. Thus, $BCDE$ is a rectangle. It follows that BE and CD are both length 12, and AC is length $36 - 12 = 24$. AB is length $36 + 12 = 48$.

Thus, triangle ABC is a 30-60-90 right triangle with BC having length $24\sqrt{3}$ which is also the length of DE. The length of the band is twice the distance that follows the arc from F to E, the line segment from E to D, and the arc from D to G. Because angle FBE is 60 degrees and angle DAG is 120 degrees, the band has length $2\left(\frac{2\cdot 12\pi}{6} + 24\sqrt{3} + \frac{2\cdot 36\pi}{3}\right) = 48\sqrt{3} + 56\pi$. The requested sum is $48 + 3 + 56 = 107$.

Problem 22

The diagram shows a parabola, a line perpendicular to the parabola's axis of symmetry, and three similar isosceles triangles each with a base on the line and vertex on the parabola. The two smaller triangles are congruent and each have one base vertex on the parabola and one base vertex shared with the larger triangle. The ratio of the height of the larger triangle to the height of the smaller triangles is $\frac{a+\sqrt{b}}{c}$ where a, b, and c are positive integers, and a and c are relatively prime. Find $a + b + c$.

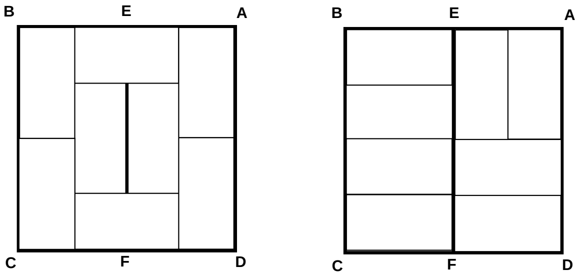


Answer: 20

Place the parabola on a coordinate plane so it is symmetric around the y axis, and the bases of the triangles are along the x axis. Since all parabolas are similar, one can assume that the points on the parabola satisfy $1 - x^2 = y$. Let the larger triangle have base length $2r$ and each smaller triangle have base length $2s$. Then the large triangle has vertices at $(\pm r, 0)$ and $(0, 1)$ while the smaller triangle to the right has vertices at $(r, 0)$, $(1, 0)$, and $(r + s, \frac{s}{r})$. Then the distance from the origin to the right most vertex of the smaller triangle is $1 = r + 2s$. The top vertex of the smaller triangle is on the parabola so it satisfies $1 - (r + s)^2 = \frac{s}{r}$. Substituting $r = 1 - 2s$ into the latter equation and simplifying yields $2s^2 - 5s + 1 = 0$ which has solution $s = \frac{5-\sqrt{17}}{4}$ implying $r = \frac{\sqrt{17}-3}{2}$. Since the triangles are similar, the ratio of the height of the large triangle to the height of the small triangles is $\frac{r}{s} = \frac{2(\sqrt{17}-3)}{5-\sqrt{17}} = \frac{2(\sqrt{17}-3)}{5-\sqrt{17}} \cdot \frac{5+\sqrt{17}}{5+\sqrt{17}} = \frac{1+\sqrt{17}}{2}$. The requested sum is 1 + 17 + 2 = 20.

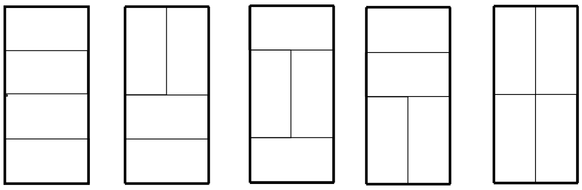
Problem 23

Square $ABCD$ has side length 4. Points E and F are the midpoints of sides AB and CD, respectively. Eight 1 by 2 rectangles are placed inside the square so that no two of the eight rectangles overlap (see diagram). If the arrangement of eight rectangles is chosen randomly, then there are relatively prime positive integers m and n so that $\frac{m}{n}$ is the probability that none of the rectangles crosses the line segment EF (as in the arrangement on the right). Find $m + n$.



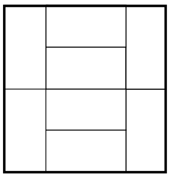
Answer: 61

First count the number of ways to place the rectangles so that none of them cross the segment EF. In these arrangements four 1 by 2 rectangles cover the 2 by 4 rectangle $AEFD$, and four 1 by 2 rectangles cover the 2 by 4 rectangle $EBCF$. Four 1 by 2 rectangles can cove a 2 by 4 rectangle in five ways:

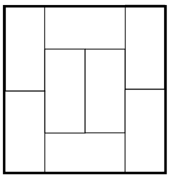


Thus, there are $5 \cdot 5 = 25$ ways to cover the 4 by 4 rectangle without crossing the segment EF.

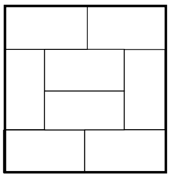
Now count the number of ways to cover the 4 by 4 rectangle while crossing the segment EF. The segment EF can be crossed by either two 1 by 2 rectangles or by four 1 by 2 rectangles. The possible arrangements are:
Thus, there are 11 ways to cover the 4 by 4 rectangle while crossing the segment EF. It follows that the probability that a randomly selected arrangement does not cross the segment EF is $\frac{25}{25+11} = \frac{25}{36}$. The requested sum is $25 + 36 = 61$.



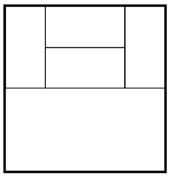
There is 1 way to cross with 4 rectangles.



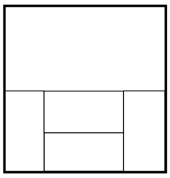
There is 1 way to cross with rectangles at the top and bottom.



There is 1 way to cross with 2 rectangles in the middle.



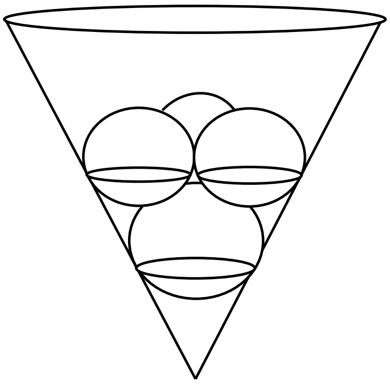
There are 4 ways to cross with 2 rectangles at the top.



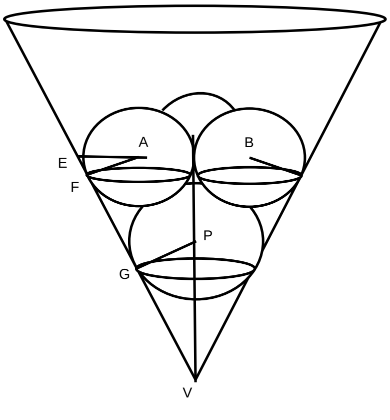
There are 4 ways to cross with 2 rectangles at the bottom.

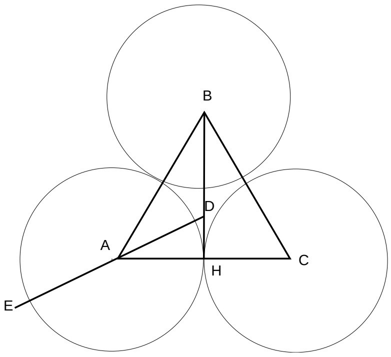
Problem 24

A right circular cone pointing downward forms an angle of 60° at its vertex. Sphere S with radius 1 is set into the cone so that it is tangent to the side of the cone. Three congruent spheres are placed in the cone on top of S so that they are all tangent to each other, to sphere S, and to the side of the cone. The radius of these congruent spheres can be written as $\frac{a+\sqrt{b}}{c}$ where a, b, and c are positive integers such that a and c are relatively prime. Find $a + b + c$.



Answer: 364





Let P be the center of the first sphere, and A, B, and C be the centers of the three congruent spheres. Let D be the circumcenter of the triangle ABC, H be the foot of the altitude of ABC on side AB, and E be the intersection of ray DA and the cone. Let F be the point where the sphere centered at A touches the cone, G be a point where sphere S touches the cone, and V be the vertex of the cone. Let r be the radius of the three congruent spheres. Triangle VPG is a 30-60-90 triangle, so segment PV has length 2. Triangle AEF is also a 30-60-90 triangle, so segment AE has length $\frac{2r}{\sqrt{3}}$.

Triangle ABC is equilateral with side length $2r$, so its altitude is $r\sqrt{3}$, and since the circumcenter D is one-third the distance from H to B, the circumradius is $\sqrt{r^2 + \frac{r^2}{3}} = \frac{2r}{\sqrt{3}}$. It follows that the length of DE is $\frac{4r}{\sqrt{3}}$, and since triangle VED is another 30-60-90 triangle, the length of DV is $4r$. The length of DP is then $4r - 2$, the length of AP is $r + 1$, and the Pythagorean Theorem gives that $\left(\frac{2r}{\sqrt{3}}\right)^2 + (4r-2)^2 = (r+1)^2$. This reduces to $49r^2 - 54r + 9 = 0$ giving solutions $r = \frac{27 \pm \sqrt{288}}{49}$. The requested sum is $27 + 288 + 49 = 364$. Note that the second answer gives the radii of three small congruent spheres which fit underneath the sphere of radius 1 rather than on top of it.

Problem 25

The polynomial $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_8x^8 + 2009x^9$ has the property that $P\left(\frac{1}{k}\right) = \frac{1}{k}$ for $k = 1, 2, 3, 4, 5, 6, 7, 8, 9$. There are relatively prime positive integers m and n such that $P\left(\frac{1}{10}\right) = \frac{m}{n}$. Find $n - 10m$.

Answer: 20090

Let $Q(x) = x^9\left(P\left(\frac{1}{x}\right) - \frac{1}{x}\right)$. This defines $Q(x)$ to be a polynomial for all $x \neq 0$, and so it can be considered to be a polynomial for all values of x. The polynomial Q is zeros at $x = 1, 2, 3, 4, 5, 6, 7, 8, 9$, and $Q(0) = 2009$. Thus, $Q(x) = -2009\frac{(x-1)(x-2)(x-3)\cdots(x-9)}{9!}$. Since $P(\frac{1}{x}) = \frac{Q(x)}{x^9} + \frac{1}{x}$, the value of $P(\frac{1}{10})$ is $\frac{Q(10)}{10^9} + \frac{1}{10} = \frac{-2009}{10^9} + \frac{1}{10} = \frac{10^8 - 2009}{10^9}$. The requested value is $n - 10m = 10^9 - 10(10^8 - 2009) = 20090$.