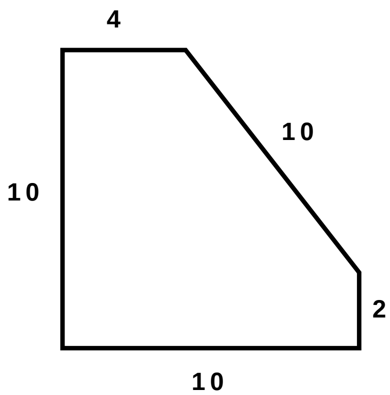
PURPLE COMET MATH MEET April 2009

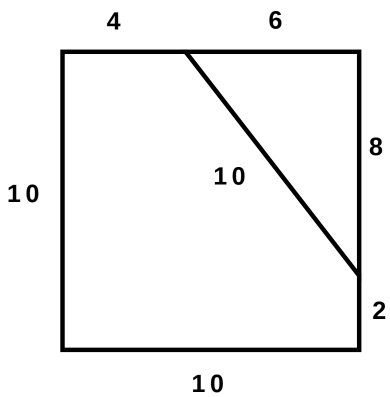
MIDDLE SCHOOL - SOLUTIONS

Problem 1

The pentagon below has three right angles. Find its area.



Answer: 76



By adding a $6:8:10$ right triangle, the pentagon becomes a square as shown. Thus, the requested area is the area of the square minus the area of the triangle which equals $10^2 - \frac{6\cdot 8}{2} = 100 - 24 = 76$.

Problem 2

Let $p_1 = 2,\ p_2 = 3,\ p_3 = 5,\ \ldots$ be the sequence of prime numbers. Find the least positive even integer n so that $p_1 + p_2 + p_3 + \cdots + p_n$ is not prime.

Answer: 8

When n is 2, the sum is 2 + 3 = 5 which is prime. When n is 4, the sum is 2 + 3 + 5 + 7 = 17 which is prime. When n is 6, the sum is 2 + 3 + 5 + 7 + 11 + 13 = 41 which is prime. But when n is 8, the sum is 2 + 3 + 5 + 7 + 11 + 13 + 17 + 19 = 77 which is 7 times 11, so it is not prime. Thus, the answer is 8.

Problem 3

The Purple Comet*!* Math Meet runs from April 27 through May 3, so the sum of the calendar dates for these seven days is 27 + 28 + 29 + 30 + 1 + 2 + 3 = 120. What is the largest sum of the calendar dates for seven consecutive Fridays occurring at any time in any year?

Answer: 142

Seven consecutive Fridays either include four Fridays from a single month or five Fridays from a single month. If five Fridays occur in a month, the largest sum of the those five dates would be 3 + 10 + 17 + 24 + 31 = 85. If two of the seven Fridays are in the following month, they would have a date sum of 7 + 14 = 21. No choice of Fridays in the first month would allow the sum of the two dates from the second month to be greater, so the largest total possible when there are five Fridays one month followed by two in the next month is 85 + 21 = 106. If five Fridays occur in one month with two Fridays in the previous month, the two Fridays in the previous month could have a total of at most 20 + 27 = 47 giving a total of 85 + 47 = 132. If five Fridays occur in a month with one Friday in the previous month and one in the following month, the largest total possible is 27 + 85 + 7 = 119.

If four Fridays occur in a month, the largest sum of those four dates would be 10 + 17 + 24 + 31 = 82. The largest sum for three Fridays in the next month is 7 + 14 + 21 = 42. No choice of dates in the first month can result in a higher sum of three dates in the second month, so the largest total possible when there are four Fridays in the first month is 82 + 42 = 124. If there are four Fridays in a month along with three Fridays in the previous month, the largest sum of those four dates from the second month would be 7 + 14 + 21 + 28 = 70. This occurs exactly when one gets the largest possible sum of three dates from the first month, 17 + 24 + 31 = 72. The total for the seven Fridays is then 72 + 70 = 142. If one or two Fridays are moved from the first month to the month following the month with four Fridays, either the 17 is replace by a 7 or the 17 and 24 are replaced by 7 and 14. In either case, the sum is smaller. Thus, the largest sum occurs when the first Friday of the seven consecutive Fridays falls on the 17-th day of a 31 day month giving the total 142.

Problem 4

John, Paul, George, and Ringo baked a circular pie. Each cut a piece that was a sector of the circle. John took one-third of the whole pie. Paul took one-fourth of the whole pie. George took one-fifth of the whole pie. Ringo took one-sixth of the whole pie. At the end the pie had one sector remaining. Find the measure in degrees of the angle formed by this remaining sector.

Answer: 18

The total number of degrees removed by the four out of the 360 degree circle was $360\left(\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}\right)=$ $120+90+72+60=342$. Thus, the number of degrees remaining is $360-342=18$.

Problem 5

A train car held 6000 pounds of mud which was 88 percent water. Then the train car sat in the sun, and some of the water evaporated so that now the mud is only 82 percent water. How many pounds does the mud weigh now?

Answer: 4000

The original mud was 88 percent water, so it was 12 percent dirt, or $6000(.12)=720$ pounds dirt. The mud now weighs m and is 82 percent water, so it is 18 percent dirt. Thus, $m(.18)=720$ and
$m=\frac{720}{.18}=\frac{72000}{18}=4000$ pounds.

Problem 6

Find n so that $20^{2009}=10^{2000}\cdot 40^{9}\cdot 2^{n}$.

Answer: 1991

The given equation is equivalent to $2^{4018}\cdot 5^{2009}=2^{2000}\cdot 5^{2000}\cdot 2^{27}\cdot 5^{9}\cdot 2^{n}$. The same number of factors of 5 appear on each side of this equation. For the number of factors of 2 to be the same, one needs
$4018=2000+27+n$. Thus, $n=4018-2000-27=1991$.

Problem 7

How many distinct four letter arrangements can be formed by rearranging the letters found in the word **FLUFFY**? For example, FLYF and ULFY are two possible arrangements.

Answer: 72

Consider the arrangements that contain four distinct letters (that is, with no repeated F). There are $4!=$ 24 such arrangements. Then consider the arrangements that contain two F's. To select such an arrangement, one must choose the two letters to go along with the two F's in one of $\binom{3}{2} = 3$ ways and then choose a rearrangement of the two chosen letters with the two repeated F's in one of $\frac{4!}{2!} = 12$ ways. Thus, there are $3 \cdot 12 = 36$ of these arrangements. Finally, consider the arrangements that contain all three F's. To select such an arrangement, one must choose a fourth letter in one of 3 ways and then choose a rearrangement of the chosen letter with the three repeated F's in one of $\frac{4!}{3!} = 4$ ways. Thus, there are $3 \cdot 4 = 12$ of these arrangements. The total number of arrangement is, therefore, $24 + 36 + 12 = 72$.

Problem 8

Find the number of non-congruent scalene triangles whose sides all have integral length, and the longest side has length 11.

Answer: 20

Let the sides of such a triangle be $a < b < c = 11$. Then, by the triangle inequality, $7 \leq b \leq 10$ and $c - b < a < b$. Thus, as b decreases by 1, the range of a decreases by 2. For b = 10 we have $2 \leq a \leq 9$, hence the number of triangles is $8 + 6 + 4 + 2 = 20$.

Problem 9

One plant is now 44 centimeters tall and will grow at a rate of 3 centimeters every 2 years. A second plant is now 80 centimeters tall and will grow at a rate of 5 centimeters every 6 years. In how many years will the plants be the same height?

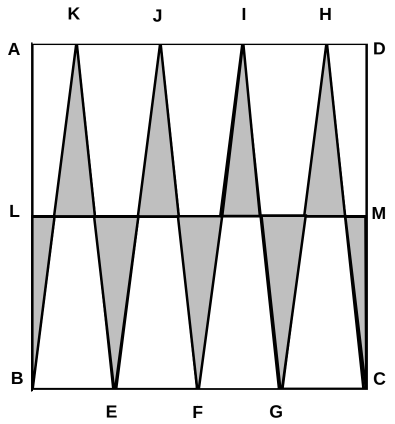
Answer: 54

The first plant grows at a rate of $\frac{3}{2}$ centimeters per year, so after t years it has reached a height of $44 + \frac{3}{2}t$. The second plant grows at a rate of $\frac{5}{6}$ centimeters per year, so after t years it has reached a height of $80 + \frac{5}{6}t$. These two heights are equal when $44 + \frac{3}{2}t = 80 + \frac{5}{6}t$. This simplifies to $\left(\frac{3}{2} - \frac{5}{6}\right) t = 80 - 44$ or $\frac{2}{3}t = 36$ and $t = \frac{3}{2} \cdot 36 = 54$.

Alternatively, the first plant grows 9 centimeters every 6 years, so it gains on the second plant at a rate of 4 centimeters every 6 years. It needs to grow $80 - 44 = 36 = 9 \cdot 4$ more centimeters than the second plant to catch up to the second plant. Thus, it must grow for $9 \cdot 6 = 54$ years to catch up.

Problem 10

The diagram shows a 20 by 20 square $ABCD$. The points E, F, and G are equally spaced on side BC. The points H, I, J, and K on side DA are placed so that the triangles BKE, EJF, FIG, and GHC are isosceles. Points L and M are midpoints of the sides AB and CD, respectively. Find the total area of the shaded regions.



Answer: 100

Each of the shaded triangles has area equal to half of the length of its base times its height. Each of these triangles has height 10, and the sum of the bases of the triangles is 20. Thus, the combined areas of the shaded triangles is $\frac{10 \cdot 20}{2} = 100$. Note that the answer remains the same even if the points are not equally spaced, and the triangles are not isosceles.

Problem 11

Aisha went shopping. At the first store she spent 40 percent of her money plus four dollars. At the second store she spent 50 percent of her remaining money plus 5 dollars. At the third store she spent 60 percent of her remaining money plus six dollars. When Aisha was done shopping at the three stores, she had two dollars left. How many dollars did she have with her when she started shopping?

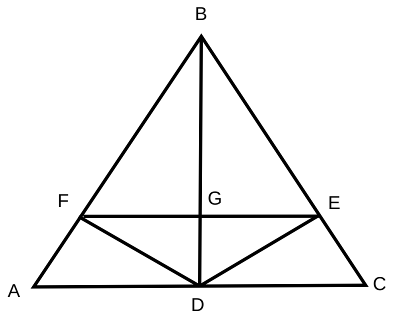
Answer: 90

Working backwards, at the third store Aisha spent 60 percent of her money plus six dollars and ended up with two dollars. Thus, she entered the third store with $\frac{6+2}{1-.6} = 20$ dollars. At the second store Aisha spent 50 percent of her money plus 5 dollars and ended up with 20 dollars. Thus, she entered the store with $\frac{20+5}{1-.5} = 50$ dollars. At the first store Aisha spent 40 percent of her money plus 4 dollars and ended up with 50 dollars. Thus, she entered the store with $\frac{50+4}{1-.4} = 90$ dollars.

Problem 12

In isosceles triangle ABC sides AB and BC have length 125 while side AC has length 150. Point D is the midpoint of side AC. E is on side BC so that BC and DE are perpendicular. Similarly, F is on side AB so that AB and DF are perpendicular. Find the area of triangle DEF.

Answer: 1728



Let G be the intersection of EF and BD as in the diagram. Note that both triangles ADB and CDB are $3:4:5$ right triangles. Each of the triangles CED, DEB, EGB, AFD, DFB, and FGB are right triangles that share an acute angle with either ADB or CDB, and, thus, they are all similar $3:4:5$ right triangles. From this similarity, side BD in triangle CDB has length $125 \cdot \frac{4}{5} = 100$. Then side BE in triangle DEB has length $100 \cdot \frac{4}{5} = 80$. Similarly, GB in triangle EGB has length $80 \cdot \frac{4}{5} = 64$, GE has length $80 \cdot \frac{3}{5} = 48$, and GD has length $|BD| - |BG| = 100 - 64 = 36$. The area of triangle DEF is the product of the lengths of GE and GD or $48 \cdot 36 = 1728$.

Problem 13

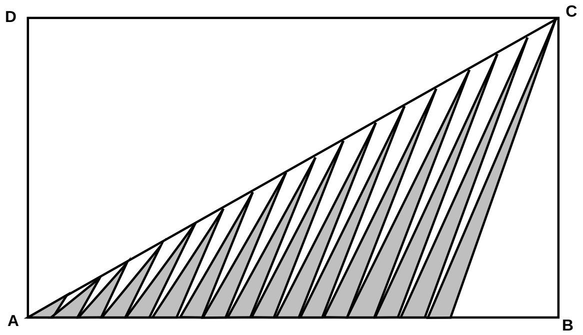
How many subsets of the set $\{1, 2, 3, ..., 12\}$ contain exactly one or two prime numbers.

Answer: 1920

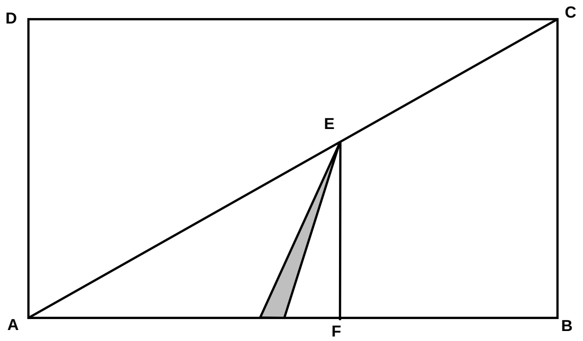
The prime numbers from set $A = \{1, 2, 3, ..., 12\}$ form the subset $B = \{2, 3, 5, 7, 11\}$ while $C = A \setminus B = \{1, 4, 6, 8, 9, 10, 12\}$. Set B has 5 elements while set C contains 7 elements. A subset of A that contains exactly one or two prime numbers is the union of a subset of B of size one or two with any subset of set C. Set B has 5 subsets of size one and $\binom{5}{2} = 10$ subsets of size two. Set C has $2^7 = 128$ subsets. It follows that there are $(5 + 10) \cdot 128 = 1920$ acceptable sets.

Problem 14

Rectangle $ABCD$ measures 70 by 40. Eighteen points (including A and C) are marked on the diagonal AC dividing the diagonal into 17 congruent pieces. Twenty-two points (including A and B) are marked on the side AB dividing the side into 21 congruent pieces. Seventeen non-overlapping triangles are constructed as shown. Each triangle has two vertices that are two of these adjacent marked points on the side of the rectangle, and one vertex that is one of the marked points along the diagonal of the rectangle. Only the left 17 of the 21 congruent pieces along the side of the rectangle are used as bases of these triangles. Find the sum of the areas of these 17 triangles.



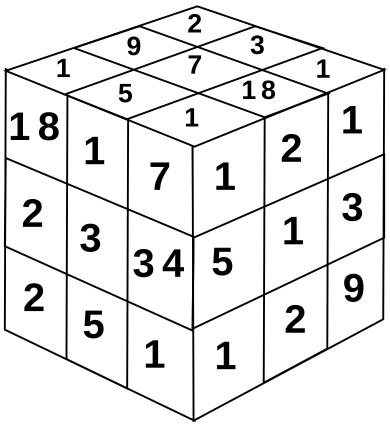
Answer: 600



Let the diagonal of the rectangle have length d. Consider the k-th triangle from the left as shown with vertex E along the diagonal. Let F be the point on AB so that EF is perpendicular to AB. Then the height of the triangle is the length of EF which is, by similar triangles, $|AE| \cdot \frac{|BC|}{|AC|} = \frac{kd}{17}\frac{40}{d} = \frac{40k}{17}$. The area of the triangle is then given by $\frac{1}{2}\frac{70}{21}\frac{40k}{17} = \frac{40 \cdot 70}{2 \cdot 21 \cdot 17}k$. The sum of the areas of the 17 triangles is $\sum_{k=1}^{17} \frac{40 \cdot 70}{2 \cdot 21 \cdot 17}k = \frac{40 \cdot 70}{2 \cdot 21 \cdot 17}\frac{17 \cdot 18}{2} = 600$. Note that we assumed that side AB was the side that measures 70 rather than 40, but the answer is the same if AB is the side that measures 40.

Problem 15

We have twenty-seven 1 by 1 cubes. Each face of every cube is marked with a natural number so that two opposite faces (top and bottom, front and back, left and right) are always marked with an even number and an odd number where the even number is twice that of the odd number. The twenty-seven cubes are put together to form one 3 by 3 cube as shown. When two cubes are placed face-to-face, adjoining faces are always marked with an odd number and an even number where the even number is one greater than the odd number. Find the sum of all of the numbers on all of the faces of all the 1 by 1 cubes.



Answer: 1377

Suppose one face of a 1 by 1 cube is on the outside of the 3 by 3 cube, and that face is marked with the odd number n. Then the face opposite that face on the 1 by 1 cube is marked with $2n$. The face on the next cube adjacent to the face with $2n$ must be marked with $2n-1$. Its opposite face must be marked $4n-2$. The face adjacent to it must be marked $4n-3$, and the face opposite this face on its 1 by 1 cube must be marked $8n-6$. For example, if a 1 appears on the side of the big cube, then $8 \cdot 1 - 6 = 2$ will appear on the opposite side of the big cube. The sum of the six numbers starting with n and ending on the opposite side of the big cube with $8n-6$ is $n + (2n) + (2n-1) + (4n-2) + (4n-3) + (8n-6) = 21n - 12$. The diagram shows fourteen faces marked with either a 1 or 2 = 8(1) - 6, five faces marked with either a 3 or an 18 = 8(3) - 6, four faces marked with either a 5 or a 34 = 8(5) - 6, two faces marked with a 7, and two faces marked with a 9. Thus, the 27 odd numbers appearing on the outside of the 3 by 3 cube add up to $14 \cdot 1 + 5 \cdot 3 + 4 \cdot 5 + 2 \cdot 7 + 2 \cdot 9 = 81$. For each of these odd numbers n on the outside of the 3 by 3 cube, there are six other sides of 1 by 1 cubes adding up to $21n - 12$. It follows that the sum of all the numbers marked on all the sides of 1 by 1 cubes is $21 \cdot 81 - 27 \cdot 12 = 81(21-4) = 81 \cdot 17 = 1377$.